

Triad representation of the Chern-Simons state in quantum gravity

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We investigate a triad representation of the Chern-Simons state of quantum gravity with a non-vanishing cosmological constant. It is shown that the Chern-Simons state, which is a well-known exact wavefunctional within the Ashtekar theory, can be transformed to the real triad representation by means of a suitably generalized Fourier transformation, yielding a complex integral representation for the corresponding state in the triad variables. It is found that topologically inequivalent choices for the complex integration contour give rise to linearly independent wavefunctionals in the triad representation, which all arise from the *one* Chern-Simons state in the Ashtekar variables. For a suitable choice of the normalization factor, these states turn out to be gauge-invariant under arbitrary, even topologically non-trivial gauge-transformations. Explicit analytical expressions for the wavefunctionals in the triad representation can be obtained in several interesting asymptotic parameter regimes, and the associated semiclassical 4-geometries are discussed. In restriction to Bianchi-type homogeneous 3-metrics, we compare our results with earlier discussions of homogeneous cosmological models. Moreover, we define an inner product on the Hilbert space of quantum gravity, and choose a natural gauge-condition fixing the time-gauge. With respect to this particular inner product, the Chern-Simons state of quantum gravity turns out to be a *non-normalizable* wavefunctional.

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I. INTRODUCTION

After four decades of vigorous research, a consistent quantization of general relativity remains as one of the most fundamental problems in theoretical physics. Aside from string theory [1,2], a promising approach to this problem is provided by a *canonical* quantization of gravity. Since early attempts in the sixties [3,4], canonical quantum gravity enjoyed a renaissance after Ashtekar's discovery of complex spin-connection variables [5,6], which replaced the metric variables used so far. The new *Ashtekar representation* of general relativity turned out to be closely related to a Yang-Mills theory of a local $SO(3)$ -gauge-group [5], and therefore many ideas and concepts known from Yang-Mills theory could be carried over to the theory of gravity. In particular, the *loop representation*, which had just been investigated within Yang-Mills theory [7], furnished yet another representation of general relativity [5,8,9], and, moreover, a remarkable connection between gravity and knot theory [8,10]. Later on, the loop representation of general relativity advanced to a mathematically rigorous theory within the framework of discretized models of gravity, the so-called quantum spin-networks [11,12].

As one crucial advantage of the Ashtekar representation the constraint operators of quantum gravity took a polynomial form in the new spin-connection variables, and explicit solutions were found. Among the different quantum states discussed so far [13,14], the *Chern-Simons state* [15,16] played an outstanding role, since it was the only wavefunctional with a well-defined semiclassical limit.¹ A loop representation of the Chern-Simons state was investigated, and turned out to be closely related to the Kauffman-bracket [19]. Moreover, this particular state was found to make an obvious connection between quantum gravity and topological field theory [19,20].

However, a physical interpretation of the Chern-Simons state within the Ashtekar representation implied several problems, which arose from the *reality conditions* underlying Ashtekar's complex theory of gravity [5]. Different *real* versions of Ashtekar's theory were suggested [21–23], but the corresponding quantum constraint equations turned out to be non-polynomial, lacking the Chern-Simons state as a solution.

¹ Strictly speaking, this is only true for a non-vanishing cosmological constant, where deSitter-like 4-geometries are described by the semiclassical Chern-Simons state [14,17]. The case of a vanishing cosmological constant has been investigated by Ezawa in [18], where it turned out, that the semiclassical 4-geometries will in general suffer from different pathologies.

Amazingly, a rather natural way to circumvent the problems associated with Ashtekar's reality conditions has never been investigated: If we would be able to transform the Chern-Simons state from the Ashtekar to the metric representation, the geometrical meaning of the fundamental variables would be obvious, and no further reality conditions would be needed. In addition, also questions concerning the normalizability of the Chern-Simons state are much easier to discuss in the real metric variables, than in the complex Ashtekar spin-connection variables. It is therefore interesting to find an explicit transformation connecting these two representations, and to study the Chern-Simons state in the metric representation.

Recently, we examined this problem in the framework of the *homogeneous* Bianchi-type IX model [24–26]. As an intermediate step, we introduced the *triad representation* of general relativity, which is trivially connected to the metric representation we were interested in. Then it turned out that the Chern-Simons state in the Ashtekar representation can be transformed to the triad variables by a suitably generalized Fourier transformation. Topologically inequivalent choices for the *complex* integration contour in the Fourier integral gave rise to different, linearly independent quantum states in the triad representation, which all arose from the *one* Chern-Simons state in the Ashtekar variables. We found explicit integral representations for the corresponding states in the triad variables, and gave semiclassical interpretations of the wavefunctions in different asymptotic parameter regimes.

In the present paper, we now want to push these results for the homogeneous model a big step further, and will ask for the corresponding form of the *inhomogeneous* Chern-Simons state in the triad representation. For technical reasons, we will restrict ourselves to model Universes, where the spatial hypersurfaces of constant time are compact and without boundaries, but of arbitrary topology. In order to recover the Chern-Simons state as a quantum state of gravity, we should allow for a non-vanishing cosmological constant, which, by the way, is in complete agreement with current cosmological data [27,28].

The rest of this paper is organized as follows: In section II we define our notation and start from the metric representation of classical general relativity. We introduce the triad and the Ashtekar variables, and give new representations of the constraint observables in terms of a single tensor density, which is closely related to the curvature of the Ashtekar spin-connection. A canonical quantization of the theory is performed in section III. Choosing a particular factor ordering for the constraint operators of quantum gravity, we discuss the corresponding operator algebra, and show that it closes without any quantum corrections. The transformation connecting the Ashtekar and the triad representation is explained in detail, and is then used to derive a functional integral representation for the Chern-Simons state in the triad representation. In section IV we study several asymptotic expansions of this functional integral in some physically interesting parameter regimes. In particular, we are interested in the semiclassical form of the Chern-Simons state, which then will allow for a discussion of the semiclassical 4-geometries. A separate subsection IV B 1 is dedicated to the behavior of the Chern-Simons state under large, topologically non-trivial $SO(3)$ -gauge-transformations. The value of the Chern-Simons state on Bianchi-type homogeneous 3-manifolds is computed and compared with earlier results obtained within the framework of homogeneous models. In section V we define an inner product on the Hilbert space of quantum gravity, which is gauge-fixed with respect to the time-redefinition-invariance, and examine the normalizability of the Chern-Simons state. Finally, we summarize our conclusions in section VI. Three appendices deal with certain technical details. In appendix A, we discuss the solvability of the saddle-point equations, which determine the semiclassical Chern-Simons state, and show how the solutions of these equations correspond to *divergence-free* triads in the limit of a vanishing cosmological constant. In appendix B, then five divergence-free triads are calculated for homogeneous Bianchi-type IX metrics, and the corresponding values of the Chern-Simons state are given. In order to comment on possible boundary conditions satisfied by the Chern-Simons state, a further appendix C deals with the asymptotic behavior of particular semiclassical 4-geometries, which arise for a special class of initial 3-metrics.

II. TRIAD REPRESENTATION AND ASHTEKAR VARIABLES

In order to set the stage and to define our notation let us briefly recall the ADM Hamiltonian formulation of general relativity [3,29,30] in terms of the densitized inverse triad \tilde{e}^i_a and its canonically conjugate momentum p_{ia} . This will be called the triad representation for short [5,16,22,31,32].

The most commonly used form of the ADM formulation [3] employs as generalized coordinates the metric tensor h_{ij} on a family of space-like 3-manifolds foliating space-time. Alternatively one may also employ the inverse metric tensor h^{ij} with $h^{ij}h_{jk} = \delta^i_k$, or, what will be done here, the densitized inverse metric

$$\tilde{a}^{ij} = h h^{ij} \quad (2.1)$$

with $h = \det(h_{ij})$.² Then the canonically conjugate momenta π_{ij} , which form a tensor density of weight -1 , become

$$\pi_{ij} = \frac{\delta L}{\delta \dot{a}^{ij}} = \frac{1}{\gamma \sqrt{h}} K_{ij} , \quad (2.2)$$

where $\gamma = 16\pi G$ is a convenient abbreviation containing Newton's constant G , and K_{ij} is the usual extrinsic curvature describing the embedding of the 3-manifold in space-time. The quantity L in (2.2) is the Lagrangian defined by the Einstein-Hilbert action [29,30], in which we include a cosmological term with a cosmological constant Λ . This choice of variables implies a symplectic structure on phase-space defined by the Poisson-brackets

$$\begin{aligned} \left\{ \tilde{a}^{ij}(x), \pi_{k\ell}(y) \right\} &= \frac{1}{2} \left(\delta_k^i \delta_\ell^j + \delta_\ell^i \delta_k^j \right) \delta^3(x-y) , \\ \left\{ \tilde{a}^{ij}(x), \tilde{a}^{k\ell}(y) \right\} &= 0 = \left\{ \pi_{ij}(x), \pi_{k\ell}(y) \right\} . \end{aligned} \quad (2.3)$$

Indices i, j will be raised and lowered by h^{ij} and its inverse. In order to move on to the triad representation let us introduce the densitized inverse triad \tilde{e}^i_a via

$$\tilde{e}^i_a \cdot \tilde{e}^j_a = \tilde{a}^{ij} , \quad (2.4)$$

and define an enlarged phase-space by introducing canonically conjugate momenta p_{ia} of the \tilde{e}^i_a with Poisson-brackets

$$\begin{aligned} \left\{ \tilde{e}^i_a(x), p_{jb}(y) \right\} &= \delta_j^i \delta_{ab} \delta^3(x-y) , \\ \left\{ p_{ia}(x), p_{jb}(y) \right\} &= 0 . \end{aligned} \quad (2.5)$$

In the following we shall also make use of the triad 1-forms e_{ia} and the triad vectors $e^i_a = \tilde{e}^i_a / \sqrt{h}$. In order to guarantee that (2.3) is compatible with (2.4), (2.5), we relate π_{ij} to p_{ja} via

$$\pi_{ij} = \frac{1}{2\sqrt{h}} e_{ia} p_{ja} , \quad (2.6)$$

which serves to satisfy the first of eqs. (2.3). Furthermore we introduce the three additional constraints

$$\tilde{\mathcal{J}}_a := \varepsilon_{abc} \tilde{e}^i_b p_{ic} \stackrel{!}{=} 0 . \quad (2.7)$$

Here the Levi-Cevita tensor ε_{abc} is defined by

$$\varepsilon_{abc} := \varepsilon(e_{ia}) \cdot [abc] , \quad (2.8)$$

where $\varepsilon(e_{ia}) \in \{\pm 1\}$ measures the orientation of the triad e_{ia} , and $[abc]$ is the totally antisymmetric Levi-Cevita symbol normalized such that $[123] = +1$. On the constraint hypersurface defined by (2.7) the quantity π_{ij} defined by (2.6) is easily checked to be symmetric in i, j as required by (2.2) and to satisfy the last of eqs. (2.3).

The ADM-Hamiltonian [29,30]

$$H^{\text{ADM}} = \int d^3x \left(N \tilde{\mathcal{H}}_0^{\text{ADM}} + N^i \tilde{\mathcal{H}}_i \right) \quad (2.9)$$

with Lagrangian parameters N, N^i and constraints $\tilde{\mathcal{H}}_0^{\text{ADM}}, \tilde{\mathcal{H}}_i$ given in terms of \tilde{a}^{ij} , π_{ij} is easily rewritten in terms of the triad representation using eqs. (2.4), (2.6). This yields (cf. [5,16,22,32])

²Here and in the following densities of positive weight are denoted by an upper, and densities of negative weight by a lower tilde.

$$\tilde{\mathcal{H}}_0^{\text{ADM}} = -\frac{\gamma}{4} e_{ia} \tilde{\varepsilon}^{ijk} \varepsilon_{abc} p_{jb} p_{kc} + \frac{1}{\gamma} e_{ia} \tilde{\varepsilon}^{ijk} F_{jka} + \frac{2\Lambda}{\gamma} \sqrt{h} ,$$

$$\tilde{\mathcal{H}}_i = \partial_j (\tilde{e}^j_a p_{ia}) - \tilde{e}^j_a \partial_i p_{ja} , \quad (2.10)$$

where $\tilde{\varepsilon}^{ijk}$ is the spatial Levi-Cevita tensor density,³ and $F_{jka} = \partial_j \omega_{ka} - \partial_k \omega_{ja} + \varepsilon_{abc} \omega_{jb} \omega_{kc}$ is the curvature of the Riemannian spin-connection $\omega_{ia} = -\frac{1}{2} \varepsilon_{abc} e_{jb} \nabla_i e^j_c$. The additional constraints (2.7) must of course be added to the Hamiltonian (2.9) with new Lagrangian parameters Ω_a .

The introduction of the complex Ashtekar variables [5,6,33]

$$\mathcal{A}_{ia} = \omega_{ia} \pm \frac{i\gamma}{2} p_{ia} \quad (2.11)$$

instead of the canonical momenta p_{ia} is now convenient in order to simplify the constraints. In the framework of this paper we shall use the variables \mathcal{A}_{ia} just as auxiliary quantities. In eq. (2.11) either “+” or “-” may be chosen, but we will keep this option open by using both signs together. The two choices are classically equivalent, but lead to inequivalent quantizations in the quantum theory. The Poisson-brackets in the new variables then take the form

$$\left\{ \tilde{e}^i_a(x), \mathcal{A}_{jb}(y) \right\} = \pm \frac{i\gamma}{2} \delta^i_j \delta_{ab} \delta^3(x-y) , \quad (2.12)$$

$$\left\{ \mathcal{A}_{ia}(x), \mathcal{A}_{jb}(y) \right\} = 0 . \quad (2.13)$$

The second of these relations follows from the fact that the Riemannian spin-connection ω_{ia} can be expressed as [16]

$$\omega_{ia} = \frac{\delta\phi}{\delta\tilde{e}^i_a} \quad (2.14)$$

with

$$\phi := -\frac{1}{2} \int d^3x \tilde{\varepsilon}^{ijk} e_{ia} \partial_j e_{ka} . \quad (2.15)$$

Employing \mathcal{A}_{ia} as a new and complex spin-connection it is convenient to use also its associated curvature

$$\mathcal{F}_{ija} = \partial_i \mathcal{A}_{ja} - \partial_j \mathcal{A}_{ia} + \varepsilon_{abc} \mathcal{A}_{ib} \mathcal{A}_{jc} . \quad (2.16)$$

Then the constraints take the more pleasing form (cf. [5,16,22])

$$\tilde{\mathcal{H}}_0^{\text{ADM}} \equiv \tilde{\mathcal{H}}_0 \mp i \partial_i (e^i_a \tilde{\mathcal{J}}_a) \stackrel{!}{=} 0 , \quad (2.17)$$

with

$$\tilde{\mathcal{H}}_0 = \frac{1}{\gamma} e_{ia} \left[\tilde{\varepsilon}^{ijk} \mathcal{F}_{jka} + \frac{2}{3} \Lambda \tilde{e}^i_a \right] , \quad (2.18)$$

$$\tilde{\mathcal{H}}_i \equiv \mp \frac{2i}{\gamma} \left[\tilde{e}^j_a \partial_j \mathcal{A}_{ia} - \tilde{e}^j_a \partial_i \mathcal{A}_{ja} + \mathcal{A}_{ia} \partial_j \tilde{e}^j_a \right] \stackrel{!}{=} 0 , \quad (2.19)$$

$$\tilde{\mathcal{J}}_a \equiv \pm \frac{2i}{\gamma} \left[\partial_i \tilde{e}^i_a + \varepsilon_{abc} \tilde{e}^i_c \mathcal{A}_{ib} \right] \stackrel{!}{=} 0 , \quad (2.20)$$

and the Hamiltonian

³ With our definition of ε_{abc} in (2.8) the spatial Levi-Cevita tensor density is naturally obtained as $\tilde{\varepsilon}^{ijk} = \sqrt{h} \varepsilon_{abc} e^i_a e^j_b e^k_c$.

$$H = \int d^3x \left(N \tilde{\mathcal{H}}_0 + N^i \tilde{\mathcal{H}}_i + \Omega_a \tilde{\mathcal{J}}_a \right) , \quad (2.21)$$

endowed with the symplectic structure (2.12), (2.13), is dynamically equivalent to the ADM-Hamiltonian (2.9). In fact, as long as $\Lambda \neq 0$, the constraints (2.18) - (2.20) can all be expressed in terms of the single tensor density $\tilde{\mathcal{G}}_{\Lambda,a}^i$ defined by

$$\tilde{\mathcal{G}}_{\Lambda,a}^i = \frac{1}{2} \tilde{\varepsilon}^{ijk} \mathcal{F}_{jka} + \frac{1}{3} \Lambda \tilde{e}_a^i , \quad (2.22)$$

namely

$$\tilde{\mathcal{H}}_0 \equiv \frac{2}{\gamma} e_{ia} \tilde{\mathcal{G}}_{\Lambda,a}^i \stackrel{!}{=} 0 , \quad (2.23)$$

$$\tilde{\mathcal{H}}_i = \pm \frac{2i}{\gamma} \varepsilon_{ijk} \tilde{e}_a^j \tilde{\mathcal{G}}_{\Lambda,a}^k - \mathcal{A}_{ia} \tilde{\mathcal{J}}_a \stackrel{!}{=} 0 , \quad (2.24)$$

$$\tilde{\mathcal{J}}_a = \pm \frac{6i}{\gamma \Lambda} \mathcal{D}_i \tilde{\mathcal{G}}_{\Lambda,a}^i \stackrel{!}{=} 0 , \quad (2.25)$$

where \mathcal{D}_i is the covariant derivative with respect to the connection \mathcal{A}_{ia} . For $\Lambda = 0$ the relation of the constraint $\tilde{\mathcal{J}}_a$ with $\tilde{\mathcal{G}}_{\Lambda,a}^i$ is lost. A simple way to satisfy all the constraints (2.23)-(2.25) for $\Lambda \neq 0$ is to restrict the phase space by the nine conditions

$$\tilde{\mathcal{G}}_{\Lambda,a}^i \stackrel{!}{=} 0 . \quad (2.26)$$

Eqs. (2.26) are more restrictive than the seven eqs. (2.23)-(2.25) which they imply, i.e. we can only hope to get special solutions in this manner. Remarkably, eqs. (2.26), if imposed as initial conditions, remain satisfied for all times under the time evolution generated by the Hamiltonian (2.21). This follows from the Poisson-brackets

$$\left\{ \int d^3x N^i \tilde{\mathcal{H}}_i , \int d^3y \lambda_{ja} \tilde{\mathcal{G}}_{\Lambda,a}^j \right\} = \int d^3z (N^i \partial_i \lambda_{ja} + \lambda_{ia} \partial_j N^i) \tilde{\mathcal{G}}_{\Lambda,a}^j , \quad (2.27)$$

$$\left\{ \int d^3x \Omega_a \tilde{\mathcal{J}}_a , \int d^3y \lambda_{jb} \tilde{\mathcal{G}}_{\Lambda,b}^j \right\} = \int d^3z \Omega_a \varepsilon_{abc} \lambda_{jb} \tilde{\mathcal{G}}_{\Lambda,c}^j , \quad (2.28)$$

$$\left\{ \int d^3x N \tilde{\mathcal{H}}_0 , \int d^3y \lambda_{ja} \tilde{\mathcal{G}}_{\Lambda,a}^j \right\} = \pm \frac{i}{2} \int d^3z \frac{N}{\sqrt{h}} (e_{ia} e_{jb} - 2 e_{ib} e_{ja}) \tilde{\varepsilon}^{jkl} \mathcal{D}_k \lambda_{lb} \tilde{\mathcal{G}}_{\Lambda,a}^i , \quad (2.29)$$

which may be verified with some labor using eqs. (2.22) and (2.23)-(2.25). They imply that on the subspace $\tilde{\mathcal{G}}_{\Lambda,a}^i = 0$ of phase-space

$$\left\{ H , \tilde{\mathcal{G}}_{\Lambda,a}^i \right\} = 0 , \quad (2.30)$$

i.e. this subspace is conserved.

The equations (2.26) bear a superficial formal similarity to Einstein's field equations

$$G_{\Lambda,\nu}^\mu := G^\mu{}_\nu + \Lambda \delta_\nu^\mu = 0 \quad (2.31)$$

in 4 space-time dimensions ($\mu, \nu = 0, 1, 2, 3$) with the 4-dimensional Einstein-tensor $G^\mu{}_\nu$ satisfying the Bianchi-identity

$$\nabla_\mu G^\mu{}_\nu \equiv 0 \quad (2.32)$$

and also $\nabla_\mu G_{\Lambda,\nu}^\mu = 0$, because the affine connection satisfies the metric postulate. Since $\tilde{\mathcal{G}}_{\Lambda,a}^i$ similarly decomposes in a curvature part satisfying a Bianchi-identity

$$\mathcal{D}_i (\tilde{\varepsilon}^{ijk} \mathcal{F}_{jka}) \equiv 0 , \quad (2.33)$$

and a cosmological term proportional to Λ it is a three-dimensional analog of $G_{\Lambda,\nu}^\mu$. The analogy extends even to $\mathcal{D}_i \tilde{\mathcal{G}}_{\Lambda,a}^i = 0$, which holds due to the Bianchi-identity but requires in addition for the constraint (2.20). However, it has to be kept in mind that the spin-connection \mathcal{A}_{ia} and the densitized inverse triad \tilde{e}_a^i in $\tilde{\mathcal{G}}_{\Lambda,a}^i$ are still *independent* variables. The equations (2.26) therefore are not a closed set of field equations on the spatial manifolds.

III. QUANTIZATION

Canonical quantization in the triad representation is achieved by imposing the commutation relations

$$[\tilde{e}_a^i(x), p_{jb}(y)] = i\hbar \delta_j^i \delta_{ab} \delta^3(x-y) \quad (3.1)$$

and representing $p_{ia}(x)$ as

$$p_{ia} = \frac{\hbar}{i} \frac{\delta}{\delta \tilde{e}_a^i(x)} . \quad (3.2)$$

This implies for the \mathcal{A}_{ia} the representation

$$\mathcal{A}_{ia}(x) = \omega_{ia}(x) \pm \frac{\gamma\hbar}{2} \frac{\delta}{\delta \tilde{e}_a^i(x)} , \quad (3.3)$$

where $\omega_{ia}(x)$, given by (2.14), (2.15) is a functional of $\tilde{e}_b^j(y)$ and a diagonal operator in this representation. We now have to choose a special factor ordering in the constraint operators $\tilde{\mathcal{J}}_a$, $\tilde{\mathcal{H}}_i$ and $\tilde{\mathcal{H}}_0$. It turns out that $\tilde{\mathcal{J}}_a$ does not suffer from an ordering ambiguity. We choose the factor ordering in $\tilde{\mathcal{H}}_0$ and $\tilde{\mathcal{H}}_i$ as given in (2.18) and (2.19) in order to achieve closure of the algebra of the generators. Explicitly, the generators are then given by eqs. (2.18)-(2.20) or eqs. (2.23)-(2.25) with the ordering of \tilde{e}_a^i , \mathcal{A}_{jb} given there. The algebra of the infinitesimal generators is obtained as⁴

$$\left[\int d^3x \xi^i \tilde{\mathcal{H}}_i , \int d^3y \varphi_a \tilde{\mathcal{J}}_a \right] = i\hbar \int d^3z (\xi^i \partial_i \varphi_a) \tilde{\mathcal{J}}_a , \quad (3.4)$$

$$\left[\int d^3x \xi^i \tilde{\mathcal{H}}_i , \int d^3y \eta^j \tilde{\mathcal{H}}_j \right] = i\hbar \int d^3z (\xi^i \partial_i \eta^j - \eta^i \partial_i \xi^j) \tilde{\mathcal{H}}_j , \quad (3.5)$$

$$\left[\int d^3x \varphi_a \tilde{\mathcal{J}}_a , \int d^3y \psi_b \tilde{\mathcal{J}}_b \right] = i\hbar \int d^3z \varepsilon_{abc} \varphi_a \psi_b \tilde{\mathcal{J}}_c , \quad (3.6)$$

$$\left[\int d^3x \varphi_a \tilde{\mathcal{J}}_a , \int d^3y N \tilde{\mathcal{H}}_0 \right] = 0 , \quad (3.7)$$

$$\left[\int d^3x \xi^i \tilde{\mathcal{H}}_i , \int d^3y N \tilde{\mathcal{H}}_0 \right] = i\hbar \int d^3z (\xi^i \partial_i N) \tilde{\mathcal{H}}_0 , \quad (3.8)$$

$$\left[\int d^3x N \tilde{\mathcal{H}}_0 , \int d^3y M \tilde{\mathcal{H}}_0 \right] = i\hbar \int d^3z (N \partial_i M - M \partial_i N) h^{ij} (\tilde{\mathcal{H}}_j + \mathcal{A}_{ja} \tilde{\mathcal{J}}_a) . \quad (3.9)$$

On the right hand side of these equations all generators appear on the right, which means that the algebra closes, at least formally (i.e. in the absence of any regularization procedure), without any quantum corrections.

Following Dirac [35], physical states $\Psi[\tilde{e}_a^i]$ must satisfy

$$\tilde{\mathcal{J}}_a \Psi[\tilde{e}_a^i] \stackrel{!}{=} 0 \quad \text{Lorentz invariance} , \quad (3.10)$$

$$\tilde{\mathcal{H}}_i \Psi[\tilde{e}_a^i] \stackrel{!}{=} 0 \quad \text{diffeomorphism invariance} , \quad (3.11)$$

$$\tilde{\mathcal{H}}_0 \Psi[\tilde{e}_a^i] \stackrel{!}{=} 0 \quad \text{time-redefinition invariance} . \quad (3.12)$$

⁴The algebra of the constraint operators has been discussed intensively in the literature, see e.g. [5,16,34]. The factor ordering and the corresponding operator algebra considered here are in agreement with Ashtekar's results in [5].

Moreover, since the Lorentz constraint (3.10) guarantees only invariance under local $SO(3)$ -gauge-transformations of the triad \tilde{e}^i_a , while the full symmetry group is given by $O(3)$, we further have to impose a discrete, global parity requirement

$$\mathcal{P} \Psi[\tilde{e}^i_a] := \Psi[-\tilde{e}^i_a] \stackrel{!}{=} +\Psi[\tilde{e}^i_a] , \quad (3.13)$$

where \mathcal{P} denotes the parity operator acting on functionals of the triad.

As in the classical theory, the constraints (3.10)-(3.12) on physical states are all satisfied if the stronger conditions

$$\tilde{\mathcal{G}}^i_{\Lambda,a} \Psi[\tilde{e}^i_a] \stackrel{!}{=} 0 \quad (3.14)$$

hold, where $\tilde{\mathcal{G}}^i_{\Lambda,a}$ is the tensor density defined by eqs. (2.22), (2.16) in terms of the operators \tilde{e}^i_a and \mathcal{A}_{ia} given by eqs. (2.4), (2.11). Remarkably, the quantum operators $\tilde{\mathcal{G}}^i_{\Lambda,a}$ turn out to commute among themselves. It can be seen from eqs. (2.23)-(2.25), which must now be read as operator equations, that eqs. (3.10)-(3.12) are implied by (3.14). The subspace of physical states satisfying (3.14) is the quantum version of the invariant subspace of classical phase-space defined by eqs. (2.26).

To find the solutions of eqs. (3.14) it is useful to proceed in two steps. First, it is convenient to perform a similarity transformation (cf. [16])

$$\Psi = \exp \left[\mp \frac{2}{\gamma \hbar} \phi \right] \cdot \Psi' , \quad (3.15)$$

where ϕ was defined in eq. (2.15). Under this transformation, the operators \mathcal{A}_{ia} according to (3.3) transform like

$$\exp \left[\pm \frac{2}{\gamma \hbar} \phi \right] \cdot \mathcal{A}_{ia} \cdot \exp \left[\mp \frac{2}{\gamma \hbar} \phi \right] = \pm \frac{\gamma \hbar}{2} \frac{\delta}{\delta \tilde{e}^i_a} , \quad (3.16)$$

and eq. (3.14) becomes explicitly

$$\left[\tilde{\varepsilon}^{imn} \left(\pm \gamma \hbar \partial_m \frac{\delta}{\delta \tilde{e}^n_a} + \frac{\gamma^2 \hbar^2}{4} \varepsilon_{abc} \frac{\delta^2}{\delta \tilde{e}^m_b \delta \tilde{e}^n_c} \right) + \frac{2\Lambda}{3} \tilde{e}^i_a \right] \Psi' = 0 . \quad (3.17)$$

As a second step, we now consider a representation of $\Psi'[\tilde{e}^i_a]$ by a generalized Fourier integral

$$\Psi'[\tilde{e}^i_a] = \int_{\Gamma} \mathcal{D}^9[\mathcal{A}_{ia}] \exp \left[\pm \frac{2}{\gamma \hbar} \int d^3x \tilde{e}^i_a \mathcal{A}_{ia} \right] \cdot \tilde{\Psi}[\mathcal{A}_{ia}] \quad (3.18)$$

where the complex integration manifold Γ is chosen in such a way that partial integrations with respect to \mathcal{A}_{ia} are permitted without any boundary terms. Besides these restrictions, Γ may be chosen arbitrarily to guarantee the existence of the functional integral (3.18) (cf. discussions of the homogeneous Bianchi IX model [24,26]). Different choices of Γ within these restrictions, which cannot be deformed into each other continuously without crossing a singularity of the integrand, will, in general, correspond to different solutions. Under the transformation (3.18) the fundamental operators \mathcal{A}_{ia} , \tilde{e}^i_a transform like

$$\tilde{e}^i_a \cdot \Psi' \mapsto \mp \frac{\gamma \hbar}{2} \frac{\delta \tilde{\Psi}}{\delta \mathcal{A}_{ia}} \quad , \quad \frac{\delta \Psi'}{\delta \tilde{e}^i_a} \mapsto \pm \frac{2}{\gamma \hbar} \mathcal{A}_{ia} \cdot \tilde{\Psi} \quad , \quad (3.19)$$

and equation (3.17) becomes

$$\left[\tilde{\varepsilon}^{ijk} \mathcal{F}_{jka} \mp \frac{\gamma \hbar \Lambda}{3} \frac{\delta}{\delta \mathcal{A}_{ia}} \right] \tilde{\Psi} = 0 . \quad (3.20)$$

Up to a normalization factor \mathcal{N} , the unique solution of (3.20) is the Chern-Simons state (cf. [16])

$$\tilde{\Psi}_{\text{CS}}[\mathcal{A}_{ia}] = \mathcal{N} \exp \left[\pm \frac{3}{\gamma \hbar \Lambda} \mathcal{S}_{\text{CS}}[\mathcal{A}_{ia}] \right] \quad (3.21)$$

with the Chern-Simons functional

$$\mathcal{S}_{\text{CS}}[\mathcal{A}_{ia}] = \int d^3x \, \varepsilon^{ijk} \left(\mathcal{A}_{ia} \partial_j \mathcal{A}_{ka} + \frac{1}{3} \varepsilon_{abc} \mathcal{A}_{ia} \mathcal{A}_{jb} \mathcal{A}_{kc} \right) . \quad (3.22)$$

In the \tilde{e}^i_a -representation the corresponding wavefunctional is given by

$$\Psi_{\text{CS}}[\tilde{e}^i_a] = \mathcal{N} \int_{\Gamma} \mathcal{D}^9[\mathcal{A}_{ia}] \exp \left[\pm \frac{1}{\gamma \hbar} \left(\int d^3x \, \varepsilon^{ijk} e_{ia} \mathcal{D}_j e_{ka} + \frac{3}{\Lambda} \mathcal{S}_{\text{CS}}[\mathcal{A}_{ia}] \right) \right] . \quad (3.23)$$

The state (3.23) is obviously diffeomorphism-invariant, and it is also gauge-invariant under sufficiently small gauge-transformations (i.e. those which are continuously connected to the identical transformation):⁵ The contribution from the similarity transformation (3.16) and the Fourier term from (3.18) fit perfectly together to give the first gauge-invariant term in the exponent of (3.23), while the second term proportional to \mathcal{S}_{CS} is a well-known gauge-invariant functional. The wavefunctional $\Psi_{\text{CS}}[\tilde{e}^i_a]$ given in (3.23) further turns out to be parity invariant, as it was required by the condition (3.13).

However, for a trivial choice of the prefactor \mathcal{N} in (3.23) the state $\Psi_{\text{CS}}[\tilde{e}^i_a]$ *fails* to be invariant under *large* gauge-transformations of the triad, since the Chern-Simons functional in (3.23) transforms non-trivially under such transformations (cf. [5,36]). At this point it is helpful to notice that the prefactor \mathcal{N} in (3.23), underlying the only restriction

$$\frac{\delta \mathcal{N}}{\delta \tilde{e}^i_a} \stackrel{!}{=} 0 , \quad (3.24)$$

is just required to be constant under *infinitesimal* variations of \tilde{e}^i_a , while it may still depend on topological invariants of the triad. In section IV B 1 we will make use of this remarkable freedom, choosing the normalization factor \mathcal{N} in such a way that the state $\Psi_{\text{CS}}[\tilde{e}^i_a]$ becomes invariant even under large gauge-transformations of the triad with a non-trivial winding number.

Unfortunately, the integration manifold Γ in eq. (3.23) can not be given explicitly, but we will argue that several topologically inequivalent choices for Γ do exist, which give rise to linearly independent quantum states $\Psi_{\text{CS}}[\tilde{e}^i_a]$. These different states in the \tilde{e}^i_a -representation arise all from the *one* Chern-Simons state in the \mathcal{A}_{ia} -representation, a phenomenon which is well-known from discussions of the homogeneous Bianchi IX model in earlier papers [24,26]. Together these states span the subspace of physical states corresponding to the invariant subspace of phase-space defined classically by $\tilde{\mathcal{G}}_{\Lambda,a}^i = 0$.

IV. ASYMPTOTIC EXPANSIONS OF THE CHERN-SIMONS STATE

Since the functional integral occurring in the \tilde{e}^i_a -representation of the Chern-Simons state (3.23) is too complicated to be performed analytically, we will restrict ourselves to an asymptotic evaluation of the wavefunctional (3.23) in several interesting parameter regimes. The possible different asymptotic regimes can be displayed by rewriting the Chern-Simons state (3.23) in dimensionless quantities. Therefore, we introduce the three fundamental length-scales of the theory, namely the Planck-scale

$$a_{\text{Pl}} := \sqrt{\gamma \hbar} , \quad (4.1)$$

the cosmological scale parameter

$$a_{\text{cos}} := \left(\int d^3x \, \sqrt{\hbar} \right)^{1/3} \quad (4.2)$$

and a third length-scale, which is associated with the cosmological constant Λ :

$$a_{\Lambda} := \sqrt{\frac{3}{\Lambda}} . \quad (4.3)$$

⁵Here and in the following, we shall refer to the $SO(3)$ -gauge-invariance just as “gauge-invariance” for short. The diffeomorphism- and the time-redefinition-invariance, which are of course inherent gauge-symmetries of the theory as well, will allways be mentioned separately.

These three length-scales give rise to the definition of two dimensionless parameters, for example

$$\kappa := \left(\frac{a_{\text{cos}}}{a_{\Lambda}} \right)^2 = \frac{\Lambda}{3} a_{\text{cos}}^2 \quad , \quad \mu := \left(\frac{a_{\text{cos}}}{a_{\text{Pl}}} \right)^2 = \frac{a_{\text{cos}}^2}{\gamma \hbar} \quad . \quad (4.4)$$

Moreover, we may rescale the triad fields with the help of the cosmological scale parameter a_{cos} to arrive at dimensionless field variables denoted by a prime:⁶

$$e'_{ia} = a_{\text{cos}}^{-1} e_{ia} \quad , \quad \tilde{e}'^i_a = a_{\text{cos}}^{-2} \tilde{e}^i_a \quad , \quad \sqrt{h'} = a_{\text{cos}}^{-3} \sqrt{h} \quad . \quad (4.5)$$

Making use of eqs. (4.1)-(4.5) the Chern-Simons state (3.23) reduces to the form

$$\Psi_{\text{CS}}[\tilde{e}^i_a] = \mathcal{N} \int_{\Gamma} \mathcal{D}^9[\mathcal{A}_{ia}] \exp \left[\pm \mu F \right] \quad , \quad (4.6)$$

where the exponent F is defined by

$$F := \int d^3x \tilde{\varepsilon}^{ijk} e'_{ia} \mathcal{D}_j e'_{ka} + \frac{1}{\kappa} \mathcal{S}_{\text{CS}}[\mathcal{A}_{ia}] \quad . \quad (4.7)$$

A. The semiclassical limit $\mu \rightarrow \infty$

Because of the Gaussian saddle-point form of (4.6) with respect to the parameter μ it is natural to study the limit $\mu \rightarrow \infty$ first. This limit corresponds to the regime $a_{\text{cos}} \gg a_{\text{Pl}}$, and also to the formal limit $\hbar \rightarrow 0$ (cf. eq. (4.4)), so we shall refer to it as the *semiclassical* limit for short. In the limit $\mu \rightarrow \infty$ the asymptotic form of the integral (4.6) becomes in leading order of μ

$$\Psi_{\text{CS}}[\tilde{e}^i_a] \stackrel{\mu \rightarrow \infty}{\propto} \mathcal{N} \left| \frac{\mu \delta^2 F}{\delta \mathcal{A}_{ia}(x) \delta \mathcal{A}_{jb}(y)} \right|^{-\frac{1}{2}} \cdot \exp \left[\pm \mu F \right] \quad , \quad (4.8)$$

where an infinite prefactor in (4.8) has been omitted. The asymptotic expression (4.8) has to be evaluated at a saddle-point of the exponent F with respect to \mathcal{A}_{ia} , which is obtained by solving the saddle-point equations

$$\frac{\delta F}{\delta \mathcal{A}_{ia}} = 2 \tilde{e}'^i_a + \frac{1}{\kappa} \tilde{\varepsilon}^{ijk} \mathcal{F}_{jka} \stackrel{!}{=} 0 \quad . \quad (4.9)$$

The equations (4.9) more explicitly take the form

$$\tilde{\varepsilon}^{ijk} \left(\partial_j \mathcal{A}_{ka} + \frac{1}{2} \varepsilon_{abc} \mathcal{A}_{jb} \mathcal{A}_{kc} \right) = -\frac{\Lambda}{3} \tilde{e}^i_a \quad , \quad (4.10)$$

and coincide with the classical equations $\tilde{\mathcal{G}}^i_{\Lambda,a} = 0$ as they should, since the latter constitute the classical limit of the gravitational Chern-Simons state. The saddle-point equations (4.10) must be read as determining implicitly the complex spin-connection \mathcal{A}_{ia} for any given real triad \tilde{e}^i_a , for which we wish to evaluate $\Psi_{\text{CS}}[\tilde{e}^i_a]$. Since \tilde{e}^i_a carries information about the coordinate system and the local $SO(3)$ -gauge-degrees of freedom, the solutions \mathcal{A}_{ia} of (4.10) for a *given* triad \tilde{e}^i_a have *no* further gauge-freedom. This is why we expect a discrete, finite set of solutions \mathcal{A}_{ia} of (4.10) for a fixed triad \tilde{e}^i_a . A detailed mathematical discussion of the solvability properties of the semiclassical saddle-point equation (4.10) will be given in appendix A 1.

For a fixed triad \tilde{e}^i_a the number of the different gauge-fields \mathcal{A}_{ia} solving (4.10) will depend on the topology of the spatial manifold \mathcal{M}_3 : For example, if \mathcal{M}_3 has the topology of the 3-sphere S^3 , five distinct solutions \mathcal{A}_{ia} of the corresponding saddle-point equations are found for spatially *homogeneous* 3-manifolds, which are described by the Bianchi IX model (cf. [24,26]). It follows from the arguments given in appendix A 1, that this number of saddle-points is preserved under sufficiently small inhomogeneous perturbations of the triad \tilde{e}^i_a . We therefore find *five* physically

⁶ By definition, the Ashtekar variables \mathcal{A}_{ia} carry no dimension and need not to be rescaled.

inequivalent solutions \mathcal{A}_{ia} in this case. If we consider manifolds \mathcal{M}_3 with the topology of the 3-torus T^3 , the subset of homogeneous manifolds is described by the Bianchi I model, restricting the number of independent solutions \mathcal{A}_{ia} of (4.10) to be *two*, as in this homogeneous model. Considering other topologies of \mathcal{M}_3 , the number of inequivalent saddle-points will differ further. However, we will see in subsection IV B that, for *any* given topology of the spatial 3-manifold \mathcal{M}_3 , the number of distinct saddle-points \mathcal{A}_{ia} of (4.10) should *at least* be two.

Given a topology of \mathcal{M}_3 and a saddle-point solution \mathcal{A}_{ia} of (4.10), the evaluation of (4.8) at this saddle-point gives a possible semiclassical contribution to the Chern-Simons state $\Psi_{\text{CS}}[\tilde{e}^i_a]$ in the limit $\mu \rightarrow \infty$. It will depend on the choice of the integration contour Γ in (4.6) whether this particular saddle-point contributes to the functional integral or not. Under gauge- or coordinate- transformations of the triad \tilde{e}^i_a the fixed solution \mathcal{A}_{ia} of (4.10) transforms like a spin-connection, since (4.10) is a coordinate- and gauge-covariant equation. Consequently, the semiclassical expression (4.8) remains unchanged under (sufficiently small) gauge-transformations, as it indeed must be the case, since Ψ_{CS} , also for $\mu \rightarrow \infty$, was constructed as a coordinate- and gauge-invariant state. Therefore, we may solve the equations (4.10) in any desired gauge for \tilde{e}^i_a , fixing automatically a gauge for the solutions \mathcal{A}_{ia} .

Any *possible* saddle-point contribution (4.8) for a given saddle-point \mathcal{A}_{ia} can be chosen to become the *dominant* contribution to the functional integral in (4.6) in the limit $\mu \rightarrow \infty$ by choosing the complex integration manifold Γ suitably. So the number of linearly independent semiclassical wavefunctionals $\Psi_{\text{CS}}[\tilde{e}^i_a]$ equals the number of inequivalent saddle-points \mathcal{A}_{ia} of (4.10). This is also the number of linearly independent *exact* wavefunctionals $\Psi_{\text{CS}}[\tilde{e}^i_a]$, because the complex integration manifold Γ , constructed as a contour of steepest descent to a given saddle-point \mathcal{A}_{ia} , satisfies the requirements for Γ in eq. (3.18) and may therefore be used to define an exact wavefunctional (4.6). We conclude that the *one* Chern-Simons state (3.21) in the complex Ashtekar representation generates a discrete, finite set of linearly independent gravitational states in the real triad representation, which differ by the topology of the integration manifolds Γ connecting the two representations via (3.18). The number of the different Chern-Simons states in the \tilde{e}^i_a -representation depends on the topology of the spatial manifold \mathcal{M}_3 and should at least be *two*.

We will now try to construct explicit solutions \mathcal{A}_{ia} of the non-linear, partial differential equations (4.10). In general, analytical solutions of this complicated set of equations are not available, so we will restrict ourselves to asymptotic solutions in the two different limits $\kappa \rightarrow \infty$ and $\kappa \rightarrow 0$, which will be treated in sections IV B and IV C, respectively.

B. The limit of large scale parameter $\mu \rightarrow \infty, \kappa \rightarrow \infty$

According to our definition of the parameters μ and κ in eq. (4.4), the limit $\kappa \rightarrow \infty$ within the semiclassical limit $\mu \rightarrow \infty$ can be realized by taking the scale parameter a_{cos} of the spatial manifold sufficiently large, $a_{\text{cos}} \gg a_{\text{Pl}}, a_{\Lambda}$. In this special asymptotic regime, solutions of (4.10) can be found by inserting the ansatz

$$\mathcal{A}_{ia} \stackrel{\kappa \rightarrow \infty}{\sim} \sqrt{\kappa} c_{ia}^{(0)} + \mathcal{O}(\kappa^0) \quad (4.11)$$

into the saddle-point equations

$$\tilde{\varepsilon}^{ijk} \left(\partial_j \mathcal{A}_{ka} + \frac{1}{2} \varepsilon_{abc} \mathcal{A}_{jb} \mathcal{A}_{kc} \right) = -\kappa \tilde{e}^i_a. \quad (4.12)$$

Then we find the two solutions

$$c_{ia}^{(0)} = \pm i e'_{ia}, \quad (4.13)$$

or, equivalently,

$$\mathcal{A}_{ia} \stackrel{\kappa \rightarrow \infty}{\sim} \pm i \sqrt{\frac{\Lambda}{3}} e_{ia} + \mathcal{O}(\kappa^0). \quad (4.14)$$

We should stress that the two signs occurring in (4.13), (4.14) are *independent* of the double sign in (2.11), i.e. for both possible definitions (2.11) of the Ashtekar variables we find two independent, complex conjugate solutions \mathcal{A}_{ia} of the saddle-point equations (4.12) in the limit $\kappa \rightarrow \infty$, corresponding to two semiclassical wavefunctions via (4.8). To avoid confusion, we will discuss only one of these solutions in the following, which is obtained by choosing the upper sign in eqs. (4.13), (4.14). The corresponding results for the second solution may then be obtained at any time by a complex conjugation.

The result (4.14) can be improved by calculating the coefficients $c_{ia}^{(n)}$ of the asymptotic series

$$\mathcal{A}_{ia} \stackrel{\kappa \rightarrow \infty}{\sim} \sum_{n=0}^{\infty} c_{ia}^{(n)} \kappa^{(1-n)/2} . \quad (4.15)$$

All coefficients in (4.15) can be calculated analytically, since, in any order of κ , the non-Abelian term in (4.12) contains the unknown coefficient $c_{ia}^{(n)}$, while the non-local term in (4.12) is known from the previous orders. Consequently, the recursion equations determining $c_{ia}^{(n)}$ are just algebraic equations at each space-point, which, moreover, are linear and analytically solvable for $n > 0$. The first three terms of the series (4.15) turn out to be

$$\mathcal{A}_{ia} \stackrel{\kappa \rightarrow \infty}{\sim} \underbrace{i \sqrt{\frac{\Lambda}{3}} e_{ia}}_{\mathcal{O}(\kappa^{1/2})} + \underbrace{\omega_{ia}}_{\mathcal{O}(\kappa^0)} + \underbrace{i \sqrt{\frac{3}{\Lambda}} \left(\frac{R}{4} e_{ia} - e_{ja} R^j{}_i \right)}_{\mathcal{O}(\kappa^{-1/2})} + \mathcal{O}(\kappa^{-1}) . \quad (4.16)$$

To calculate the corresponding saddle-point contribution to the semiclassical Chern-Simons state via (4.8) we need the Gaussian prefactor and the exponent F defined in (4.7), evaluated at the saddle-point \mathcal{A}_{ia} . The asymptotic form of the Gaussian prefactor becomes in the limit $\kappa \rightarrow \infty$

$$\left| \frac{\mu \delta^2 F}{\delta \mathcal{A}_{ia}(x) \delta \mathcal{A}_{jb}(y)} \right|^{-\frac{1}{2}} \stackrel{\kappa \rightarrow \infty}{\propto} \mathbb{h}^{-3/4} , \quad (4.17)$$

with the abbreviation

$$\mathbb{h} := \prod_{x \in \mathcal{M}_3} h(x) . \quad (4.18)$$

The exponent in (4.8) for $\kappa \rightarrow \infty$ can be expanded as follows:

$$F \stackrel{\kappa \rightarrow \infty}{\sim} \frac{1}{\gamma \hbar \mu} \left[i \sqrt{\frac{3}{\Lambda}} \int d^3 x \sqrt{h} \left(\frac{4\Lambda}{3} - R \right) + \frac{3}{\Lambda} \mathcal{S}_{\text{CS}}(\omega_{ia}) \right] + \mathcal{O}(\kappa^{-3/2}) . \quad (4.19)$$

Here the contribution ϕ from the similarity transformation (3.15) has disappeared, because it precisely cancels with the contribution ω_{ia} in the asymptotic series (4.16) of \mathcal{A}_{ia} . The first term in (4.19) derives from the contributions of order $\kappa^{1/2}$ and $\kappa^{-1/2}$ to the asymptotic series of \mathcal{A}_{ia} given in (4.16). It defines a real action

$$S = \pm \frac{1}{\gamma} \sqrt{\frac{3}{\Lambda}} \int d^3 x \sqrt{h} \left(\frac{4\Lambda}{3} - R \right) , \quad (4.20)$$

giving rise to a well-defined, semiclassical time evolution. The term of order κ^{-1} in the expansion (4.16), which was not given explicitly there, because it is rather lengthy, determines the asymptotic form of the second term in (4.19), which is real-valued and therefore governs the asymptotic behavior of $|\Psi_{\text{CS}}|^2$. Surprisingly, this contribution again turns out to be a Chern-Simons functional, but with \mathcal{A}_{ia} replaced by the real Riemannian spin-connection ω_{ia} . As one can check quite easily, this functional $\mathcal{S}_{\text{CS}}[\omega_{ia}]$ has the interesting property that it is also invariant under *local* scale-transformations of the triad $e_{ia} \mapsto \exp[\zeta(x)] e_{ia}$.

Inserting the results (4.17) and (4.19) into (4.8), we find for the semiclassical Chern-Simons state in the \tilde{e}^i_a -representation

$$\Psi_{\text{CS}} \stackrel{\kappa \rightarrow \infty}{\underset{\mu \rightarrow \infty}{\propto}} \mathcal{N} \cdot \mathbb{h}^{-3/4} \cdot \exp \left[\pm \frac{1}{\gamma \hbar} \left(i \sqrt{\frac{3}{\Lambda}} \int d^3 x \sqrt{h} \left(\frac{4\Lambda}{3} - R \right) + \frac{3}{\Lambda} \mathcal{S}_{\text{CS}}[\omega_{ia}] \right) \right] , \quad (4.21)$$

where the complex conjugate solution Ψ_{CS}^* is equally possible, if we choose the second saddle-point solution in eqs. (4.13), (4.14). It is remarkable that this result is universal in the sense that it does not depend on the topology of the spatial 3-manifold \mathcal{M}_3 .

1. Large gauge-transformations

An unsatisfactory feature of the asymptotic state (4.21) is the fact that its exponent is *not* invariant under large gauge-transformations with a nonvanishing winding number: As is well-known [5,36], in general the Chern-Simons functional $\mathcal{S}_{\text{CS}}[\omega_{ia}]$ transforms inhomogeneously under local gauge-transformations of the triad,

$$e_{ia} \mapsto \Omega_{ab} e_{ib} \quad \Rightarrow \quad \mathcal{S}_{\text{CS}}[\omega_{ia}] \mapsto \mathcal{S}_{\text{CS}}[\omega_{ia}] + \frac{1}{6} I(\Omega) , \quad (4.22)$$

with $(\Omega_{ab}) = \Omega \in O(3)$ being an arbitrary rotation matrix. The quantity $I(\Omega)$ occuring in (4.22) is defined by

$$I(\Omega) := \int d^3x \tilde{\varepsilon}^{ijk} \text{Tr} [\Omega^T \partial_i \Omega \cdot \Omega^T \partial_j \Omega \cdot \Omega^T \partial_k \Omega] \quad (4.23)$$

and known as the Cartan-Maurer invariant [36]. Its value is restricted to be of the form

$$I(\Omega) = I_0 \cdot w(\Omega) , \quad (4.24)$$

where the winding number $w(\Omega)$ is an integer, and I_0 is a constant depending only on the topology of the 3-manifold \mathcal{M}_3 .

A consequence of eq. (4.22) is that the asymptotic Chern-Simons state (4.21) will not be invariant under general gauge-transformations of the triad, at least as long as we make a trivial choice for the normalization factor \mathcal{N} in (4.21). However, as we pointed out in section III, the factor \mathcal{N} does not need to be *completely* independent of the triad - it is still allowed to depend on topological invariants, such as the Cartan-Maurer invariant. This is why we are free to choose the normalization factor \mathcal{N} according to

$$\mathcal{N} \propto \exp \left[\mp \frac{I(\hat{\Omega})}{2\gamma\hbar\Lambda} \right] , \quad (4.25)$$

where $\hat{\Omega}$ is a special gauge-transformation rotating the triad e_{ia} into a *gauge-fixed* triad g_{ia} of the 3-metric $h_{ij} = e_{ia}e_{ja}$. Then the requirement (3.24) remains to be satisfied, and, in addition, the Chern-Simons state (4.21) becomes invariant under arbitrary gauge-transformations of the triad e_{ia} , since the inhomogeneous term in (4.22) is cancelled precisely by a suitable contribution from the prefactor \mathcal{N} according to (4.25). With our special choice (4.25) of the normalization factor \mathcal{N} we circumvent the definition of the so-called “ Θ -angle”, which can be introduced alternatively to solve the problem associated with large gauge-transformations [5,36]. As a special, gauge-fixed triad g_{ia} in the definition of $\hat{\Omega}$ may serve the “Einstein-triad” that can be constructed by solving the eigenvalue problem of the 3-dimensional Einstein-tensor G^i_j :⁷

$$G^i_j g^j_a = \lambda_{\bar{a}} g^i_{\bar{a}} \quad , \quad g^i_a g_{ib} = \delta_{ab} . \quad (4.26)$$

2. Restriction to Bianchi-type homogeneous 3-manifolds

It is very instructive to specialize the asymptotic state (4.21) to the case of spatially homogeneous 3-manifolds. For homogeneous manifolds of one of the nine Bianchi types, the 3-metric can be expressed in terms of *invariant* triad 1-forms $e_a = \mathbf{e}_a = e_{ia} dx^i$ as (cf. [37])

$$\mathbf{h} = \mathbf{e}_a \otimes \mathbf{e}_a \quad , \quad d\mathbf{e}_a = -\frac{1}{2} m_{ba} \varepsilon_{bcd} \mathbf{e}_c \wedge \mathbf{e}_d , \quad (4.27)$$

with a *spatially constant* structure matrix $\mathbf{m} = (m_{ab})$. We should restrict ourselves to compactified, homogeneous 3-manifolds, such that the volume

$$V = \frac{1}{6} \varepsilon_{abc} \int \mathbf{e}_a \wedge \mathbf{e}_b \wedge \mathbf{e}_c \quad (4.28)$$

⁷ Here a bar over an index indicates that *no* summation with respect to this index should be performed.

is finite. If we introduce the scale-invariant structure matrix \mathbf{M} as

$$\mathbf{M} = a_{\text{cos}} \cdot \mathbf{m} , \quad (4.29)$$

the asymptotic Chern-Simons state (4.21) takes the following value for Bianchi-type homogeneous 3-manifolds:

$$\begin{aligned} \Psi_{\text{CS}} \underset{\mu \rightarrow \infty}{\overset{\kappa \rightarrow \infty}{\propto}} \mathcal{N} \cdot \mathbb{h}^{-3/4} \cdot \exp \left[\pm \mu \left(4i \sqrt{\kappa} - \frac{i}{\sqrt{\kappa}} \left[\text{Tr} \mathbf{M}^2 - 2 \text{Tr} \mathbf{M}^T \mathbf{M} + \frac{1}{2} \text{Tr}^2 \mathbf{M} \right] \right. \right. \\ \left. \left. - \frac{1}{\kappa} \left[\text{Tr} \mathbf{M}^2 \mathbf{M}^T - \frac{1}{6} \text{Tr} \mathbf{M} (\text{Tr} \mathbf{M}^2 + 2 \text{Tr} \mathbf{M}^T \mathbf{M}) + 2 \det \mathbf{M} \right] \right) \right] . \end{aligned} \quad (4.30)$$

For homogeneous manifolds of Bianchi-type IX, the determinant of the matrix \mathbf{M} is given by $\det \mathbf{M} = 8 \mathcal{V}$, where \mathcal{V} is the dimensionless, invariant volume of the unit 3-sphere, so the matrix \mathbf{M} may be parametrized by a diagonal, traceless matrix β via

$$\mathbf{M} = 2 \sqrt[3]{\mathcal{V}} e^{2\beta} . \quad (4.31)$$

Using the identity

$$\text{Tr} e^{2\beta} \cdot \text{Tr} e^{4\beta} = \text{Tr} e^{6\beta} + \text{Tr} e^{2\beta} \cdot \text{Tr} e^{-2\beta} - 3 , \quad (4.32)$$

and introducing the rescaled parameter $\kappa' := \mathcal{V}^{-2/3} \kappa/4$, we find for Bianchi-type IX homogeneous 3-manifolds:

$$\begin{aligned} \Psi_{\text{CS}} \underset{\mu \rightarrow \infty}{\overset{\kappa \rightarrow \infty}{\propto}} \mathcal{N} \cdot \mathbb{h}^{-3/4} \cdot \exp \left[\pm \frac{24\mathcal{V}}{\gamma \hbar \Lambda} \left(4i \sqrt{\kappa'}^3 - i \sqrt{\kappa'} \left[\text{Tr} e^{-2\beta} - \frac{1}{2} \text{Tr} e^{4\beta} \right] \right. \right. \\ \left. \left. - \frac{1}{2} \left[\text{Tr} e^{6\beta} - \text{Tr} e^{2\beta} \cdot \text{Tr} e^{-2\beta} + 7 \right] \right) \right] . \end{aligned} \quad (4.33)$$

Thus, up to a quantum correction in the Gaussian prefactor, we reproduce exactly the result obtained earlier within the framework of the homogeneous Bianchi IX model in [24] (cf. eq. (5.18) there). To compare the results explicitly, we have to identify κ' with the parameter κ in [24], and to set $\gamma = 16 \pi$, $\mathcal{V} = 4 \pi^2$.

In the case of flat 3-metrics, which are of Bianchi-type I, the structure matrix \mathbf{M} turns out to vanish, and (4.30) reduces to

$$\Psi_{\text{CS}} \underset{\mu \rightarrow \infty}{\overset{\kappa \rightarrow \infty}{\propto}} \mathcal{N} \cdot \mathbb{h}^{-3/4} \cdot \exp [\pm 4i \mu \sqrt{\kappa}] = \mathcal{N} \cdot \mathbb{h}^{-3/4} \cdot \exp \left[\pm \frac{4i}{\gamma \hbar} \sqrt{\frac{\Lambda}{3}} \int d^3x \sqrt{\hbar} \right] , \quad (4.34)$$

a result, which also follows directly from (4.21) by setting $R = 0$, $\mathcal{S}_{\text{CS}}[\omega_{ia}] = 0$.

3. Semiclassical 4-geometries

Let us now ask for the semiclassical trajectories and the corresponding semiclassical 4-geometries, which are described by the state (4.21) in the limit $\mu \rightarrow \infty, \kappa \rightarrow \infty$, i.e. in the limit of large scale-parameters $a_{\text{cos}} \gg a_{\text{Pl}}, a_{\Lambda}$. Choosing the Lagrangian multipliers trivially as $N = 1, N^i = 0, \Omega_a = 0$ in (2.21), we find

$$\dot{\tilde{e}}^i_a = - \{ H , \tilde{e}^i_a \} = \pm i \tilde{\varepsilon}^{ijk} \mathcal{D}_j e_{ka} = - \frac{\gamma}{2} \tilde{\varepsilon}^{ijk} \varepsilon_{abc} p_{jb} e_{kc} , \quad (4.35)$$

where the dot denotes a derivative with respect to the classical ADM time-variable t introduced in section II. The semiclassical momentum p_{ia} is given in terms of the action (4.20) of the wavefunction (4.21) by

$$p_{ia} = \frac{\delta S}{\delta \tilde{e}^i_a} , \quad (4.36)$$

or can equivalently be extracted from the asymptotic saddle-point \mathcal{A}_{ia} according to (4.16) in connection with (2.11):

$$p_{ia} \stackrel{\kappa \rightarrow \infty}{\sim} \pm \frac{2}{\gamma} \left[\sqrt{\frac{\Lambda}{3}} e_{ia} + \sqrt{\frac{3}{\Lambda}} \left(\frac{R}{4} e_{ia} - e_{ja} R^j_i \right) \right]. \quad (4.37)$$

Thus, for large scale-parameters a_{cos} the classical evolution of the triad \tilde{e}^i_a is determined by the equation

$$\mp \dot{\tilde{e}}^i_a \stackrel{a_{\text{cos}} \rightarrow \infty}{\sim} 2 \sqrt{\frac{\Lambda}{3}} \tilde{e}^i_a + \sqrt{\frac{3}{\Lambda}} \tilde{e}^j_a G^i_j, \quad (4.38)$$

which describes a deSitter-like time-evolution in leading order a_{cos} ,

$$\tilde{e}^i_a(x, t) \stackrel{a_{\text{cos}} \rightarrow \infty}{\sim} \tilde{e}^i_{a, \infty}(x) \cdot \exp \left[\mp 2 \sqrt{\frac{\Lambda}{3}} \cdot t \right], \quad (4.39)$$

with corrections described by the second term of (4.38) containing the 3-dimensional Einstein-tensor G^i_j .

The figure 1 shows an embedding of the asymptotic 4-geometry (4.39) into a flat Minkowski space, where the time direction has been chosen according to the lower sign in eq. (4.39). As is well-known for inflationary models like the one discussed within this paper, the spatial, Riemannian 3-manifolds $(\mathcal{M}_3, \mathbf{h})(t)$ tend to homogenize in the course of time t .

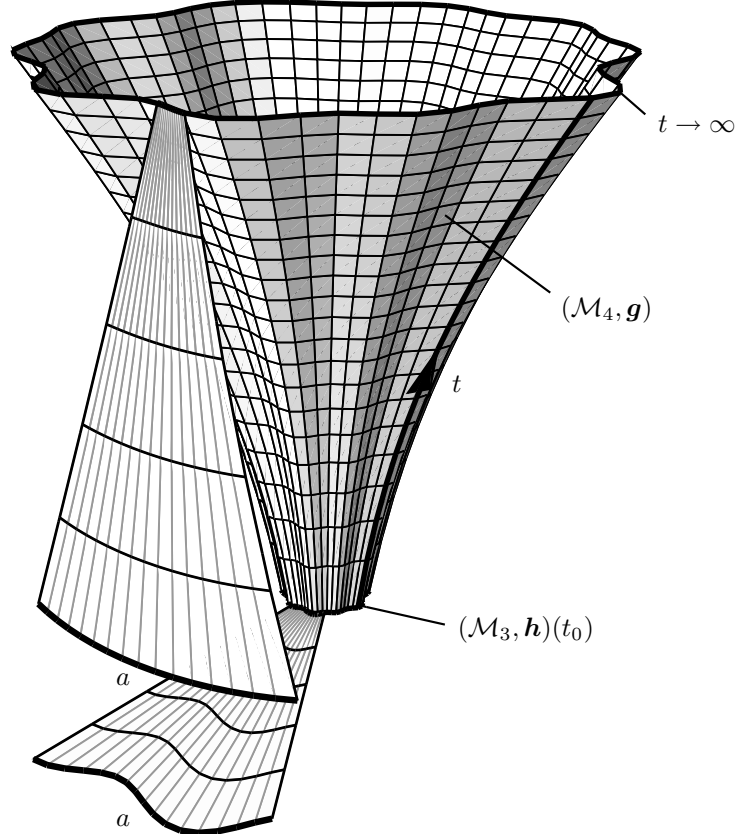


FIG. 1. Geometrical illustration of the generalized deSitter-4-geometry (4.39). The spatial 3-manifolds $(\mathcal{M}_3, \mathbf{h})(t)$ are represented by 1-dimensional curves, possible inhomogeneities are indicated by small deformations of these curves. The resulting space-time 4-manifold $(\mathcal{M}_4, \mathbf{g})$ according to (4.39) then corresponds to a 2-dimensional, Lorentzian manifold, which has been embedded into a flat, 3-dimensional Minkowski space. Portions of the marginal spatial 3-manifolds, which are of the *same* length-scale a , have been magnified to illustrate the increase in homogeneity in the course of evolution.

C. The semiclassical vacuum limit $\mu \rightarrow \infty, \kappa \rightarrow 0$

Apart from the limit $\kappa \rightarrow \infty$ there exists another asymptotic regime, where an analytical treatment of the semiclassical saddle-point equations (4.12) is tractable, namely the limit $\kappa \rightarrow 0$. By virtue of the relationships (4.4), a discussion of the Chern-Simons state (4.6) in the limit $\mu \rightarrow \infty, \kappa \rightarrow 0$ corresponds to an investigation of the asymptotic regime $a_\Lambda \gg a_{\text{cos}} \gg a_{\text{Pl}}$. This limit may be realized by considering the special case of a vanishing cosmological constant $\Lambda \rightarrow 0$ within the semiclassical limit, what will be called the semiclassical vacuum limit for short.

To find solutions of eqs. (4.12) in the limit $\kappa \rightarrow 0$ we proceed analogously to section IV B, and try a power series ansatz of the form

$$\mathcal{A}_{ia} \stackrel{\kappa \rightarrow 0}{\sim} \sum_{n=0}^{\infty} C_{ia}^{(n)} \kappa^n . \quad (4.40)$$

Then we find in the lowest order of κ

$$\varepsilon^{ijk} \left(\partial_j C_{ka}^{(0)} + \frac{1}{2} \varepsilon_{abc} C_{jb}^{(0)} C_{kc}^{(0)} \right) = 0 , \quad (4.41)$$

i.e. $C_{ia}^{(0)}$ has to be a *flat* gauge-field, which is of the general form

$$C_{ia}^{(0)} = -\frac{1}{2} \varepsilon_{abc} \Omega_{db} \partial_i \Omega_{dc} \quad \text{with} \quad \Omega \in O(3) . \quad (4.42)$$

The matrix $\Omega(x)$ is a free integration field, as long as we restrict ourselves to the leading order $\mathcal{O}(\kappa^0)$ of the saddle-point equations (4.12). However, in the next to leading order $\mathcal{O}(\kappa^1)$, we find the equations

$$\tilde{\varepsilon}^{ijk} \mathcal{D}_j^{(0)} C_{ka}^{(1)} := \tilde{\varepsilon}^{ijk} \left(\partial_j C_{ka}^{(1)} + \varepsilon_{abc} C_{jb}^{(0)} C_{kc}^{(1)} \right) \stackrel{!}{=} -\tilde{e}^i{}_a , \quad (4.43)$$

which imply additional restrictions for the coefficients $C_{ia}^{(0)}$, and thus for the matrix Ω in (4.42). These *integrability conditions* for the equations (4.43) can be obtained by operating on (4.43) with $\mathcal{D}_i^{(0)}$ from the left: Then the left hand side becomes proportional to the curvature of $C_{ia}^{(0)}$, which vanishes by virtue of eq. (4.41), and a multiplication of the resulting equations with a_{cos}^2 yields

$$\mathcal{D}_i^{(0)} \tilde{e}^i{}_a \equiv \partial_i \tilde{e}^i{}_a + \varepsilon_{abc} C_{ib}^{(0)} \tilde{e}^i{}_c \stackrel{!}{=} 0 . \quad (4.44)$$

If we insert the general solution (4.42) into (4.44), we arrive at the three integrability conditions

$$\partial_i (\Omega_{ab} \tilde{e}^i{}_b) \stackrel{!}{=} 0 , \quad (4.45)$$

which fix the integration field $\Omega(x)$ in (4.42). Moreover, the special triad fields

$$\tilde{d}^i{}_a := \Omega_{ab} \tilde{e}^i{}_b \quad (4.46)$$

with Ω chosen according to (4.45) turn out to have the geometrically interesting property of being *divergence-free*. Therefore, we may use the different possible divergence-free triads $\tilde{d}^i{}_a$ of a given Riemannian manifold $(\mathcal{M}_3, \mathbf{h})$ to parameterize the saddle-points \mathcal{A}_{ia} in the limit $\kappa \rightarrow 0$ via (4.46) and (4.42).

For a given divergence-free triad $\tilde{d}^i{}_a$, which characterizes uniquely one saddle-point solution \mathcal{A}_{ia} in the limit $\kappa \rightarrow 0$, we now wish to calculate the corresponding saddle-point contribution (4.8) to the Chern-Simons state (4.6) in the limit $\mu \rightarrow \infty, \kappa \rightarrow 0$. We first expand the exponent F defined in (4.7) for $\kappa \rightarrow 0$, and find, in particular, that the Chern-Simons functional $\mathcal{S}_{\text{CS}}[\mathcal{A}_{ia}]$ is given by

$$\mathcal{S}_{\text{CS}}[\mathcal{A}_{ia}] \stackrel{\kappa \rightarrow 0}{\sim} \frac{1}{6} I(\Omega) + \mathcal{O}(\kappa^2) . \quad (4.47)$$

Here Ω is the special rotation matrix defined in (4.45), connecting the given divergence-free triad $\tilde{d}^i{}_a$ with an arbitrary triad $\tilde{e}^i{}_a$, for which we want to evaluate $\Psi_{\text{CS}}[\tilde{e}^i{}_a]$. In (4.47) a contribution of order $\mathcal{O}(\kappa^1)$ is missing, since this term becomes proportional to the curvature of the *flat* gauge-field $C_{ia}^{(0)}$. Using eq. (4.47), the exponent F of the semiclassical Chern-Simons state takes the following form in the limit $\kappa \rightarrow 0$:

$$F \stackrel{\kappa \rightarrow 0}{\sim} \frac{I(\boldsymbol{\Omega})}{6\kappa} + \int d^3x \tilde{\varepsilon}^{ijk} d'_{ia} \partial_j d'_{ka} + \mathcal{O}(\kappa) . \quad (4.48)$$

The Cartan-Maurer invariant $I(\boldsymbol{\Omega})$ in (4.48) can be contracted with the Cartan-Maurer invariant $I(\hat{\boldsymbol{\Omega}})$ in the definition (4.25) of the normalization factor \mathcal{N} to give

$$I(\boldsymbol{\Omega}) - I(\hat{\boldsymbol{\Omega}}) \equiv I(\boldsymbol{\Omega} \cdot \hat{\boldsymbol{\Omega}}^T) =: I_0 \cdot \hat{w}[d_{ia}] . \quad (4.49)$$

Here $\hat{w}[d_{ia}]$ denotes the winding number of the divergence-free triad d_{ia} with respect to the Einstein-triad g_{ia} defined in (4.26), which is a functional of d_{ia} only: For a given divergence-free triad d_{ia} we know the 3-metric $h_{ij} = d_{ia} d_{ja}$, and therefore the Einstein-triad g_{ia} .

Inserting the results (4.49), (4.48) into (4.8), we find the following saddle-point contribution to the Chern-Simons state (4.6) in the limit $\mu \rightarrow \infty, \kappa \rightarrow 0$

$$\lim_{\kappa \rightarrow 0} \Psi_{\text{CS}} =: \Psi_{\text{vac}} \stackrel{\mu \rightarrow \infty}{\propto} \exp \left[\pm \frac{1}{\gamma \hbar} \left(\frac{I_0 \hat{w}[d_{ia}]}{2\Lambda} + \int d^3x \tilde{\varepsilon}^{ijk} d_{ia} \partial_j d_{ka} \right) \right] , \quad (4.50)$$

where the Gaussian prefactor, which contains a complicated, non-local functional determinant, has been hidden in the proportionality sign.

From the result (4.50) we can see the gauge-invariance of the semiclassical vacuum state Ψ_{vac} , since this state does not depend explicitly on the triad \tilde{e}^i_a , but only on the 3-metric $h_{ij} = e_{ia} e_{ja}$, to which we have chosen a *fixed* divergence-free triad \tilde{d}^i_a . It is remarkable that for the one unique choice (4.25) of the prefactor \mathcal{N} gauge-invariance, even under large gauge-transformations, can be achieved in both of the two quite different limits $\kappa \rightarrow \infty$ and $\kappa \rightarrow 0$.

The *existence* of divergence-free triads to a given 3-metric h_{ij} is discussed in appendix A 2. There, we also argue that in general there will even exist different, topologically inequivalent divergence-free triads, giving rise to linearly independent semiclassical vacuum states via (4.50).

1. Restriction to Bianchi-type A homogeneous 3-manifolds

We now wish to evaluate the semiclassical vacuum state (4.50) for the special case of Bianchi-type homogeneous 3-manifolds. For such manifolds, it follows directly from (4.27) that the divergence of the *invariant* triad $\vec{\tau}_a = \tau^i_a \partial_i$ can be expressed in terms of the structure matrix \mathbf{m} as

$$\vec{\nabla} \cdot \vec{\tau}_a = \frac{1}{\sqrt{h}} \partial_i \tilde{\tau}^i_a = \varepsilon_{abc} m_{bc} . \quad (4.51)$$

Consequently, the invariant triad $\vec{\tau}_a$ of Bianchi-type homogeneous 3-manifolds is divergence-free, if, and only if the structure matrix \mathbf{m} is *symmetric*, i.e. if the 3-manifold is of Bianchi-type A. If we restrict ourselves to this special class of manifolds in the following, at least *one* divergence-free triad $\vec{d}_a^{(0)} = \vec{\tau}_a$ is known, and we can calculate the corresponding value of the semiclassical vacuum state (4.50):

$$\Psi_{\text{vac}}^{(0)} \stackrel{\mu \rightarrow \infty}{\propto} \exp \left[\mp \frac{V}{\gamma \hbar} \text{Tr } \mathbf{m} \right] . \quad (4.52)$$

Here we made use of the fact that for 3-manifolds of Bianchi-type A the invariant triad $\vec{\tau}_a$ and the Einstein-triad \vec{g}_a differ only by a *spatially constant* rotation $\hat{\boldsymbol{\Omega}}$, implying a vanishing winding number $\hat{w}[\tau_{ia}] = 0$ in (4.50). A further specialization of the result (4.52) to Bianchi-type IX homogeneous manifolds gives

$$\Psi_{\text{vac}}^{(0)} \stackrel{\mu \rightarrow \infty}{\propto} \exp \left[\mp \frac{2\mathcal{V}}{\gamma \hbar} (a_1^2 + a_2^2 + a_3^2) \right] , \quad (4.53)$$

where we have introduced the three scale parameters a_b via

$$\mathbf{m} =: 2 \text{diag} \left[\frac{a_1}{a_2 a_3}, \frac{a_2}{a_3 a_1}, \frac{a_3}{a_1 a_2} \right] \quad \Rightarrow \quad V = \mathcal{V} a_1 a_2 a_3 , \quad (4.54)$$

with the same, dimensionless volume \mathcal{V} of the unit 3-sphere that already occurred in section IV B 2. The saddle-point

value (4.53) corresponds to the “wormhole-state” of the Bianchi IX model [24,26]. Within the framework of the homogeneous Bianchi IX model, four further semiclassical vacuum states are known, which, in the inhomogeneous approach of the present paper, correspond to nontrivial divergence-free triads of Bianchi-type IX manifolds via (4.50). These topologically nontrivial divergence-free triads of Bianchi-type IX metrics and the resulting values of the semiclassical vacuum state (4.50) will be discussed separately in appendix B.

As a further restriction of the state (4.52) one may consider again the case of flat Bianchi-type I manifolds, where the structure matrix \mathbf{m} , and therefore the exponent of (4.52), vanishes. Thus, for flat 3-manifolds the behavior of the semiclassical vacuum state is governed by the Gaussian prefactor, which we do not know explicitly.

2. Semiclassical 4-geometries

The semiclassical trajectories and the associated 4-geometries, which are generated by the state (4.50) in the limit $\kappa \rightarrow 0, \mu \rightarrow \infty$, can be calculated by solving the evolution equations (4.35) with the flat, semiclassical spin-connection \mathcal{A}_{ia} derived in section IV C. However, in contrast to the limit $\kappa \rightarrow \infty$ discussed in section IV B 3, we here arrive at *imaginary* evolution equations, since the semiclassical action of the wavefunctional Ψ_{vac} according to (4.50) is purely imaginary. Following Hawking [38], a geometrical interpretation may still be given in terms of an imaginary time variable $\tau := it$, converting the Lorentzian signature of the 4-dimensional space-time into a positive, Euclidian signature. Then the semiclassical evolution equations can conveniently be expressed in terms of the divergence-free triad d_{ia} , which characterizes the flat Ashtekar spin-connection \mathcal{A}_{ia} in the limit $\kappa \rightarrow 0$:

$$\frac{d}{d\tau} \tilde{d}^i{}_a = \pm \tilde{\varepsilon}^{ijk} \partial_j d_{ka} \quad \Leftrightarrow \quad \frac{d}{d\tau} d_{ia} = \mp \omega_{ia} . \quad (4.55)$$

Here ω_{ia} in the second equation is the Riemannian spin-connection of the divergence-free triad d_{ia} . Obviously, the gauge-condition $\partial_i \tilde{d}^i{}_a = 0$ remains preserved in the course of evolution, as it must be the case.

Stationary solutions of eqs. (4.55) are given by $\omega_{ia} = 0$, i.e. *flat* 3-manifolds $(\mathcal{M}_3, \mathbf{h})$. With our trivial choice of the Lagrangian multipliers $N = 1, N^i = 0$, these correspond to locally flat, positive definite semiclassical space-time manifolds $(\mathcal{M}_4, \mathbf{g})$. Further solutions of (4.55) can be constructed with help of the scaling ansatz

$$d_{ia}(x, \tau) = \mp \tau \cdot d'_{ia}(x) , \quad (4.56)$$

which implies $d'_{ia}(x) = \omega_{ia}(x)$, and therefore a simple form for the Ricci-tensor of the spatial 3-manifold:

$$R^i{}_j = \frac{2}{\tau^2} \delta^i{}_j . \quad (4.57)$$

Consequently, the spatial manifold has to be a 3-sphere with radius τ , and the 4-dimensional line element becomes

$$ds^2 = d\tau^2 + \tau^2 d\Omega_3^2 , \quad (4.58)$$

with $d\Omega_3^2$ being the line element of the unit 3-sphere. As for the stationary solutions mentioned above, the line element (4.58) describes a locally flat, positive definite 4-manifold.

Because of the nonlinearity of the evolution equations (4.55), the general behavior of the solutions is quite complicated and cannot be discussed here. However, a complete discussion of the possible semiclassical trajectories can be given within the narrow class of Bianchi-type IX homogeneous 3-manifolds, cf. [24]. There it turns out, that the semiclassical evolution governed by the invariant, divergence-free triad $\tilde{d}_a^{(0)} = \vec{\tau}_a$, which corresponds to the “wormhole-state” (4.53) via (4.50), gives rise to asymptotically *flat* 4-geometries in the limit of large scale parameters a_{cos} . Moreover, a second divergence-free triad of these Bianchi-type IX homogeneous 3-manifolds, which is given in appendix B, is known to evolve in such a way, that *compact, regular* 4-manifolds are approached in the limit of vanishing scale parameter a_{cos} .⁸

One may now ask, if such a universal behavior of the semiclassical trajectories, that can be found within the Bianchi IX model, carries over to the inhomogeneous case. Unfortunately, this does not seem to be the case: In appendix C we explicitly solve the evolution equations (4.55) for a particular class of initial 3-manifolds, and find,

⁸The semiclassical vacuum state, corresponding to this second divergence-free triad via (4.50), is the “no-boundary-state” of the Bianchi IX model.

that these solutions neither satisfy the condition of asymptotical flatness in the limit $a_{\text{cos}} \rightarrow \infty$, nor the “no-boundary” proposal suggested by Hartle and Hawking [38–40]. Thus we conclude that, in the inhomogeneous case, the semiclassical vacuum state given in (4.50) will in general *not* be subject to any specific boundary condition, like the “no-boundary” proposal or the condition of asymptotical flatness.

V. NON-NORMALIZABILITY OF THE CHERN-SIMONS STATE IN A PHYSICAL INNER PRODUCT

We now want to argue that the gravitational Chern-Simons state $\Psi_{\text{CS}}[\tilde{e}^i_a]$ according to eq. (3.23) does *not* constitute a normalizable physical state on the Hilbert space of quantum gravity. Therefore, we will derive a physical inner product on the configuration space of real triads, which we want to be gauge-fixed with respect to the time-reparametrization invariance of general relativity. In this particular inner product, we then will try to calculate the corresponding norm of the Chern-Simons state $\Psi_{\text{CS}}[\tilde{e}^i_a]$.

To derive a physical inner product within the framework of the Faddeev-Popov calculus [41,42], we first have to find a kinematical inner product, denoted by $\langle \cdot | \cdot \rangle$ in the following, with respect to which the quantum constraint operators $\tilde{\mathcal{H}}_0$, $\tilde{\mathcal{H}}_i$ and $\tilde{\mathcal{J}}_a$ are formally hermitian. Since the *complex* Hamiltonian constraint operator $\tilde{\mathcal{H}}_0$ defined in eq. (2.18) cannot be hermitian with respect to *any* inner product on the configuration space, we replace $\tilde{\mathcal{H}}_0$ by its real version $\tilde{\mathcal{H}}_0^{\text{ADM}}$ given in (2.17), with the factor ordering suggested there. With the help of the commutators (3.4)–(3.9) one can check quite easily that the algebra of $\tilde{\mathcal{H}}_0^{\text{ADM}}$, $\tilde{\mathcal{H}}_i$ and $\tilde{\mathcal{J}}_a$ still closes without any quantum corrections. However, the explicit commutators turn out to be much more complicated than the corresponding commutators of $\tilde{\mathcal{H}}_0$, $\tilde{\mathcal{H}}_i$, $\tilde{\mathcal{J}}_a$ given in (3.4)–(3.9), and will not be given here.

Since the quantum state Ψ_{CS} given in (3.23) is also annihilated by the operator $\tilde{\mathcal{H}}_0^{\text{ADM}}$, the substitution $\tilde{\mathcal{H}}_0 \mapsto \tilde{\mathcal{H}}_0^{\text{ADM}}$ has no negative consequences for the theory, but the positive effect that we can now define a kinematical inner product, with respect to which the operators $\tilde{\mathcal{H}}_0^{\text{ADM}}$, $\tilde{\mathcal{H}}_i$ and $\tilde{\mathcal{J}}_a$ are hermitian. This product turns out to be

$$\langle \Psi | \Phi \rangle = \int \mathcal{D}^9 [e_{ia}] \Psi^*[e_{ia}] \cdot \Phi[e_{ia}] , \quad (5.1)$$

where the functional integral has to be performed over all real triads $e_{ia}(x)$. While $\tilde{\mathcal{H}}_0^{\text{ADM}}$ and $\tilde{\mathcal{J}}_a$ are formally hermitian in the product (5.1), $\tilde{\mathcal{H}}_i$ is hermitian only if we take a regularization of the theory, where terms containing the singular object $(\partial_i \delta)(0)$ vanish.⁹ If we can achieve this, we have found a kinematical inner product on the configuration space of all real triads $e_{ia}(x)$, and can continue with the Faddeev-Popov calculus by choosing a gauge-condition $\tilde{\chi}[e_{ia}] = 0$ fixing the time-gauge. The corresponding physical inner product is then obtained as

$$\langle \langle \Psi | \Phi \rangle \rangle_{\text{phys}} = \langle \Psi | \delta[\tilde{\chi}] \cdot | J_{\text{H}} | | \Phi \rangle , \quad (5.2)$$

with the Faddeev-Popov functional determinant

$$J_{\text{H}} := \det \left(\frac{i}{\hbar} \left[\tilde{\mathcal{H}}_0^{\text{ADM}}(x), \tilde{\chi}(y) \right] \right) . \quad (5.3)$$

A rather natural way to fix the time-gauge is to consider 3-geometries with a given volume-form $\sqrt{h(x)}$, for which there remain only two local degrees of freedom. Therefore we assume $\tilde{v}(x)$ to be a fixed, positive scalar density of weight +1 on the spatial manifold \mathcal{M}_3 , normalized such that¹⁰

$$\int d^3x \tilde{v}(x) \stackrel{!}{=} 1 . \quad (5.4)$$

Furthermore, let a_χ be an arbitrary, positive scale parameter. Then the gauge-condition

$$\tilde{\chi} := \sqrt{h(x)} - a_\chi^3 \tilde{v}(x) \stackrel{!}{=} 0 \quad (5.5)$$

is a diffeomorphism- and $SO(3)$ -gauge-invariant equation fixing the volume-form of the 3-metric. In particular, it follows from eq. (5.5) that the length scale a_χ and the cosmological scale a_{cos} introduced in (4.2) must be equal. In

⁹Some authors argue that this should be possible, cf. Matschull [32].

¹⁰ For example, the quantity \tilde{v} may be chosen as the rescaled volume element of a maximally symmetric 3-metric on \mathcal{M}_3 .

the gauge (5.5), the physical norm associated with the inner product (5.2) obviously depends on the scale parameter a_χ and the choice of $\tilde{v}(x)$, but we can consider the limit $a_\chi \rightarrow \infty$,

$$\|\Psi\|_\infty^2 := \lim_{a_\chi \rightarrow \infty} \langle\langle \Psi | \Psi \rangle\rangle_{\text{phys}} , \quad (5.6)$$

which, in case of the Chern-Simons state $\Psi = \Psi_{\text{CS}}$, will turn out to be independent of $\tilde{v}(x)$. For an explicit calculation of (5.6), we need the Faddeev-Popov commutator occurring in (5.3), which turns out to be

$$\frac{i}{\hbar} \left[\tilde{\mathcal{H}}_0^{\text{ADM}}(x), \tilde{\chi}(y) \right] = \frac{\gamma}{4} \delta^3(x-y) \tilde{j}(x) , \quad (5.7)$$

with

$$\tilde{j}(x) := \frac{i\hbar}{2} \left[e_{ia}(x) \frac{\delta}{\delta e_{ia}(x)} + \frac{\delta}{\delta e_{ia}(x)} e_{ia}(x) \right] . \quad (5.8)$$

The Faddeev-Popov functional determinant J_{H} according to (5.3) follows as

$$J_{\text{H}} = \prod_{x \in \mathcal{M}_3} \frac{\gamma}{4} \tilde{j}(x) , \quad (5.9)$$

which, acting on the wavefunctional Ψ_{CS} , measures the space product of the current $\tilde{j}(x)$ of Ψ_{CS} in the $h(x)$ -direction of superspace. Since we are dealing with the limit $a_\chi = a_{\text{cos}} \rightarrow \infty$, the exact quantum state Ψ_{CS} given in (3.23) may be substituted by the asymptotic state (4.21) for explicit calculations. Then the current of Ψ_{CS} in the $h(x)$ -direction turns out to have the same sign at each space-point for large scale parameters $a_\chi = a_{\text{cos}}$,¹¹ so we do *not* need to take the modulus of the Faddeev-Popov determinant in (5.2), as the general calculus in [41] would prescribe. More explicitly, we find the result

$$J_{\text{H}} \cdot \Psi_{\text{CS}} \Big|_{\tilde{\chi}=0} \stackrel{a_\chi \rightarrow \infty}{\propto} \mathbb{h}^{1/2} \cdot \Psi_{\text{CS}} \Big|_{\tilde{\chi}=0} , \quad (5.10)$$

where \mathbb{h} was defined in (4.18), so the physical norm (5.6) becomes in the limit $a_\chi \rightarrow \infty$:

$$\|\Psi_{\text{CS}}\|_\infty^2 \propto \int \mathcal{D}^9[e_{ia}] \mathbb{h}^{1/2} |\Psi_{\text{CS}}|^2 \delta[\tilde{\chi}] . \quad (5.11)$$

If we now introduce the new integration variables \sqrt{h} , and eight locally scale-invariant fields β_κ , the functional integral in (5.11) becomes

$$\begin{aligned} \|\Psi_{\text{CS}}\|_\infty^2 &\propto \int \mathcal{D}[\sqrt{h}] \mathcal{D}^8[\beta_\kappa] w[\beta_\kappa] \mathbb{h}^{3/2} |\Psi_{\text{CS}}|^2 \delta[\sqrt{h} - a_{\text{cos}}^3 \tilde{v}] \\ &= \int \mathcal{D}^8[\beta_\kappa] w[\beta_\kappa] \exp \left[\pm \frac{6}{\gamma \hbar \Lambda} \hat{\mathcal{S}}_{\text{CS}}[\beta_\kappa] \right] , \end{aligned} \quad (5.12)$$

where

$$\hat{\mathcal{S}}_{\text{CS}}[\beta_\kappa] := \mathcal{S}_{\text{CS}}[\omega_{ia}] - \frac{1}{6} I(\hat{\Omega}) \quad (5.13)$$

is a locally scale-invariant functional describing the exponent of $|\Psi_{\text{CS}}|^2$ according to (4.21) and (4.25). The weight function $w[\beta_\kappa]$ occurring in (5.12) depends on the choice of the new integration variables β_κ . Since the integrand of (5.12) is locally scale-invariant, the integral is independent of the choice of $\tilde{v}(x)$ in (5.5), as announced above, so the gauge-condition $\tilde{\chi} = 0$ can be omitted in the second line of (5.12).

¹¹ This property of Ψ_{CS} in the limit $a_{\text{cos}} \rightarrow \infty$ reminds one of the Vilenkin proposal for the wavefunction of the Universe discussed in [43,44].

As a result, we find that the diffeomorphism-, gauge- and locally scale-invariant functional $\hat{\mathcal{S}}_{\text{CS}}[\beta_\kappa]$, which is closely related to the Chern-Simons functional of the Riemannian spin-connection ω_{ia} , governs the “probability”-distribution associated with the Chern-Simons state (4.21) in the limit $a_{\text{cos}} \rightarrow \infty$. Since the functional $\mathcal{S}_{\text{CS}}[\omega_{ia}]$ is obviously unbounded from above and below, we conclude that the norm (5.12) cannot be finite, even if we fix the remaining gauge-freedoms concerning the diffeomorphism- and the local $SO(3)$ -gauge-transformations.

However, we should keep in mind that the result (5.12) has been derived for a very special choice of the gauge-condition $\tilde{\chi}$ according to (5.5). Since different gauge-fixings of the Hamiltonian constraint give rise to *inequivalent* physical inner products on the Hilbert space of quantum gravity,¹² there may still exist other choices of $\tilde{\chi}$, for which the Chern-Simons state $\Psi_{\text{CS}}[\tilde{e}^i_a]$ turns out to be normalizable.

VI. DISCUSSION AND CONCLUSION

The main purpose of this paper was to derive and discuss a triad representation of the Chern-Simons state, which is a well-known exact wavefunctional of quantum gravity within Ashtekar’s theory of general relativity. In particular, we were interested in an explicit transformation connecting the real triad representation with the complex Ashtekar representation. Therefore, we first investigated this transformation on the classical level in section II. Here we also derived new representations for the constraint observables $\tilde{\mathcal{H}}_0$, $\tilde{\mathcal{H}}_i$ and $\tilde{\mathcal{J}}_a$ in terms of a single tensor-density $\tilde{\mathcal{G}}^i_{\Lambda,a}$ defined in (2.22), which is closely related to the curvature \mathcal{F}_{ija} of the Ashtekar spin-connection \mathcal{A}_{ia} .

Then, in section III, we performed a canonical quantization of the theory in the triad representation. In the particular factor ordering for the quantum constraint operators $\tilde{\mathcal{H}}_0$, $\tilde{\mathcal{H}}_i$ and $\tilde{\mathcal{J}}_a$ suggested by the equations (2.23)-(2.25) we found that the constraint algebra closes formally without any quantum corrections.

On the quantum mechanical level, the transformation from the Ashtekar- to the triad representation turned out to be given by a generalized Fourier transformation (3.18) and a subsequent similarity transformation (3.15). Here it was essential to allow for an arbitrary *complex* integration manifold Γ in the Fourier integral (3.18), restricted only by the condition that partial integrations should be permitted without getting any boundary terms.

Making use of the transformations (3.15) and (3.18), we then recovered the Chern-Simons state of quantum gravity by searching for a wavefunctional which is annihilated by $\tilde{\mathcal{G}}^i_{\Lambda,a}$. The Chern-Simons state in the triad representation turned out to be given by the complex functional integral (3.23). In our approach the Ashtekar variables played only the role of convenient auxiliary quantities. The *reality conditions* originally introduced by Ashtekar in [5] did nowhere enter explicitly, but lie hidden in the choice of the integration contour Γ for the functional integrals in (3.18) and (3.23).

We did not try to perform the complex functional integral occurring in (3.23) analytically, but restricted ourselves to semiclassical expansions of the Chern-Simons state, which were treated in section IV. Rewriting the state $\Psi_{\text{CS}}[\tilde{e}^i_a]$ in suitable dimensionless field-parameters, the functional integral turned out to be of a Gaussian saddle-point form in the semiclassical limit $\mu \rightarrow \infty$, and the semiclassical Chern-Simons state was determined by solutions of the saddle-point equations (4.10). Here it depended on the choice of the integration contour Γ , which particular saddle-points contributed to the functional integral (3.23) via (4.8). In order to prove the consistency of the semiclassical expansions, we argued for the solvability of the saddle-point equations (4.10) in a separate appendix A 1 from a mathematical point of view, where it turned out, that saddle-point solutions will exist at least under the restriction $R(x) \neq 2\Lambda$.

We were able to find explicit analytical results for the semiclassical Chern-Simons state in the two asymptotic regimes $\kappa = \Lambda a_{\text{cos}}^2/3 \rightarrow \infty$ and $\kappa \rightarrow 0$, which were discussed in sections IV B and IV C, respectively.

In the limit $\kappa \rightarrow \infty$, two different solutions of the saddle-point equations (4.10) could be found, giving rise to the linearly independent asymptotic states Ψ_{CS} and Ψ_{CS}^* given in (4.21). For a suitable choice of the normalization factor \mathcal{N} according to (4.25), these asymptotic states turned out to be invariant under arbitrary, even topologically non-trivial $SO(3)$ -gauge-transformations of the triad. In the special case of Bianchi-type homogeneous 3-metrics, we obtained the explicit result (4.30) for the value of the asymptotic Chern-Simons state (4.21), which, by a further restriction to Bianchi-type IX metrics, coincided with the corresponding result known from discussions of the homogeneous Bianchi-type IX model.

The asymptotic Chern-Simons state (4.21) in the limit $\kappa \rightarrow \infty$ gives rise to a well-defined semiclassical time-evolution, which we discussed in section IV B 3. There it turned out, that for large scale parameters a_{cos} the semi-

¹² This is a peculiarity of the Hamiltonian constraint, and in contrast to gauge-fixing procedures associated with $\tilde{\mathcal{H}}_i$ or $\tilde{\mathcal{J}}_a$, for which the Faddeev-Popov calculus guarantees a *unique* physical inner product [41,42].

classical 4-geometries associated with the Chern-Simons state are given by inhomogeneously generalized deSitter space-times.

In the limit $\kappa \rightarrow 0$, the semiclassical saddle-point contributions to the Chern-Simons state can be characterized by divergence-free triads \vec{d}_a of the Riemannian 3-manifold $(\mathcal{M}_3, \mathbf{h})$ via (4.50). Thus we had to answer the non-trivial question, whether divergence-free triads to a given 3-metric will in general *exist*, what was done in appendix A 2.

In restriction to homogeneous manifolds of Bianchi-type A, *one* divergence-free triad was explicitly known, giving rise to the result (4.52). In particular, we were able to recover the “wormhole-state” (4.53), which is a well-known vacuum state within the homogeneous Bianchi IX model. For Bianchi-type IX manifolds, four further divergence-free triads $\vec{d}_a^{(\alpha)}$, $\alpha \in \{1, 2, 3, 4\}$, were constructed in appendix B. They gave rise to four additional saddle-point contributions $\Psi_{\text{vac}}^{(\alpha)}$, $\alpha \in \{1, 2, 3, 4\}$, to the vacuum Chern-Simons state, which, however, were restricted to occur simultaneously. We concluded that, together with the “wormhole-state”, only *two* linearly independent values of the vacuum Chern-Simons state are realized for Bianchi-type IX manifolds.

Since these two values should continue to exist under sufficiently small, inhomogeneous perturbations of the 3-metric, and since also in the limit $\kappa \rightarrow \infty$ exactly *two* different values of the semiclassical Chern-Simons state were found, one may assume that the one Chern-Simons state in the Ashtekar representation corresponds to *two* linearly independent states in the triad representation.

Within the narrow class of Bianchi-type IX metrics, the semiclassical 4-geometries associated with the vacuum Chern-Simons state (4.50) are satisfying physically interesting boundary conditions, namely either the “no-boundary” condition proposed by Hartle and Hawking [38–40], or the condition of asymptotical flatness at large scale parameters a_{cos} . However, this does *not* remain true for general 3-metrics, as we have shown by exhibiting a counter-example in appendix C. We conclude that, in general, the Chern-Simons state will not satisfy the “no-boundary” condition or the condition of asymptotical flatness. Nevertheless, as we have remarked in section V, the asymptotic state (4.21) in the limit $\kappa \rightarrow \infty$ reminds one of the *Vilenkin* proposal for the wavefunction of the Universe [43,44].

In section V, we investigated the normalizability of the Chern-Simons state (3.23) in the triad representation. We defined a kinematical inner product on the Hilbert space of quantum gravity, and by performing a special gauge-fixing for the time-gauge we arrived at the physical inner product (5.2). Unfortunately, the Chern-Simons state turned out to be *non-normalizable* with respect to this particular inner product. However, as we have pointed out, there may still exist other gauge-fixing procedures (e.g. the one suggested by Smolin and Soo in [17]), which render the Chern-Simons state to be normalizable.

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APPENDIX A: ON THE SOLVABILITY OF THE SADDLE-POINT EQUATIONS

The solvability of the semiclassical saddle-point equations (4.10) is essential in order to justify the consistency of the asymptotical expansions of the Chern-Simons state discussed in section IV. Therefore, it is worth to study the solvability properties of the nonlinear, partial differential equations (4.10) from a mathematical point of view, what will be done in section A 1. Applying the results of section A 1 to the special case of a vanishing cosmological constant Λ , we will then, in section A 2, be able to prove the existence of divergence-free triads of Riemannian 3-manifolds, which determine the semiclassical vacuum state (4.50).

1. The general case $\Lambda \neq 0$

If we want to discuss the solvability of the saddle-point equations (4.10) within the theory of partial differential equations (cf. [45]), it is *not* advisable to study this problem in the particular form (4.10), since the spatial derivative operator, which is given by the curl of the gauge-field \mathcal{A}_{ia} , is known to be *non-elliptic*. However, we will show that it is possible to consider a set of second order partial differential equations instead, which will turn out to be elliptic in leading derivative order, thus allowing for solvability statements concerning the solutions \mathcal{A}_{ia} .

Let us first introduce new variables

$$\mathcal{K}_{ij} := (\omega_{ia} - \mathcal{A}_{ia}) e_{ja} \equiv \mp i K_{ji} \quad (\text{A1})$$

instead of the gauge-fields \mathcal{A}_{ia} , where e_{ia} denotes a fixed triad for which we want to solve the set of equations (4.10). Up to a Wick-rotation, the tensor \mathcal{K}_{ij} plays the role of the semiclassical extrinsic curvature tensor K_{ij} (cf. eqs. (2.2), (2.6) and (2.11)). If we rewrite the saddle-point equations (4.10) in terms of the new variables \mathcal{K}_{ij} , they become

$$\mathcal{G}_{\Lambda,j}^i := \frac{1}{\sqrt{h}} \tilde{\mathcal{G}}_{\Lambda,a}^i e_{ja} = G_{\Lambda,j}^i + {}^* \mathcal{K}^i_j - \frac{1}{\sqrt{h}} \tilde{\varepsilon}^{ik\ell} \nabla_k \mathcal{K}_{\ell j} \stackrel{!}{=} 0, \quad (\text{A2})$$

where

$${}^* \mathcal{K}^i_j := \frac{1}{2} \tilde{\varepsilon}^{ik\ell} \tilde{\varepsilon}_{jmn} \mathcal{K}_k^m \mathcal{K}_\ell^n \quad (\text{A3})$$

are the cofactors of the matrix-elements \mathcal{K}_i^j , and $G_{\Lambda,j}^i$ is the usual, 3-dimensional Einstein-tensor with a cosmological term. In analogy to (2.25), the set of equations (A2) implies the three Gauß-constraints

$$\begin{aligned} \tilde{\mathcal{J}}_a &= \pm \frac{6i}{\gamma \Lambda} \left[\nabla_j \mathcal{G}_{\Lambda,i}^j - \sqrt{h} \tilde{\varepsilon}_{ijk} \mathcal{K}_\ell^j \mathcal{G}_\Lambda^{\ell k} \right] \tilde{e}^i_a \\ &\equiv \pm \frac{2i}{\gamma} e_{ia} \tilde{\varepsilon}^{ijk} \mathcal{K}_{jk} \stackrel{!}{=} 0, \end{aligned} \quad (\text{A4})$$

which require the tensor \mathcal{K}_{ij} to be *symmetric* in i and j . Therefore, if we take \mathcal{K}_{ij} to be symmetric in the following, the Gauß-constraints (A4) are satisfied identically, and the first line of (A4) takes the form of three generalized Bianchi-identities. We thus conclude that the set of equations (A2) constitutes only *six* independent equations for the *six* fields $\mathcal{K}_{ij} = \mathcal{K}_{ji}$ we are searching for.

Beside the Gauß-constraints (A4), four further equations are implied by (A2) via (2.23) and (2.24), namely the Hamiltonian constraint

$$\tilde{\mathcal{H}}_0^{\text{ADM}} = \frac{2\sqrt{h}}{\gamma} \mathcal{G}_{\Lambda,i}^i \equiv \frac{\sqrt{h}}{\gamma} \left(\mathcal{K}^2 - \mathcal{K}_i^j \mathcal{K}_j^i + 2\Lambda - R \right) \stackrel{!}{=} 0, \quad (\text{A5})$$

and the three diffeomorphism-constraints

$$\tilde{\mathcal{H}}_i = \mp \frac{2ih}{\gamma} \tilde{\varepsilon}_{ijk} \mathcal{G}_\Lambda^{jk} \equiv \pm \frac{2i\sqrt{h}}{\gamma} \left(\nabla_j \mathcal{K}_i^j - \nabla_i \mathcal{K} \right) \stackrel{!}{=} 0, \quad (\text{A6})$$

respectively. Here \mathcal{K} in (A5) and (A6) denotes the trace of (\mathcal{K}_i^j) . Remarkably, the Hamiltonian constraint (A5) is a purely algebraical equation for \mathcal{K}_{ij} , which will be solved explicitly later on, while the diffeomorphism-constraints (A6) are *linear* equations and contain information about the *divergence* of the fields \mathcal{K}_{ij} .

Moreover, since the equations (A2) contain the curl of the fields \mathcal{K}_{ij} , eqs. (A2) and (A6) together may be used to construct a second order derivative operator similar to the Laplace-Beltrami-operator of \mathcal{K}_{ij} . Let us therefore consider the following second order differential equations

$$\Delta_{ij} := \sqrt{h} \left[\tilde{\varepsilon}_{jmn} \nabla_i \mathcal{G}_\Lambda^{mn} - \tilde{\varepsilon}_{imn} h^{mk} \nabla_k \mathcal{G}_{\Lambda,j}^n + \frac{1}{2} \tilde{\varepsilon}_{ijk} \nabla_n \mathcal{G}_\Lambda^{nk} \right] \stackrel{!}{=} 0, \quad (\text{A7})$$

which must be satisfied for solutions \mathcal{K}_{ij} of (A2). The first term in (A7) can be simplified with help of (A6), and gives in the leading derivative order the gradient of the divergence of \mathcal{K}_{ij} and, in addition, the Hessian of \mathcal{K} . Making use of eqs. (A2), the second term in (A7) contributes the curl of the curl of \mathcal{K}_{ij} , i.e. taking the first two terms in (A7) together, we arrive at

$$\Delta_{ij} = \nabla_i \nabla_j \mathcal{K} - \Delta \mathcal{K}_{ij} + \mathcal{O}(\nabla_i \mathcal{K}_{jk}) \quad (\text{A8})$$

in leading derivative order. By virtue of eqs. (A4), the third term in (A7) contains only first order derivatives of \mathcal{K}_{ij} . It has been added to obtain simple expressions for the trace and the antisymmetric part of Δ_{ij} , which are given by

$$h^{ij} \Delta_{ij} \equiv 0, \quad \tilde{\varepsilon}^{ijk} \Delta_{jk} \equiv \frac{\gamma}{2} h^{ij} \nabla_j \tilde{\mathcal{H}}_0^{\text{ADM}}. \quad (\text{A9})$$

Instead of solving the nine equations (A7), we may therefore consider the six equations

$$\Delta_{(ij)} := \frac{1}{2} (\Delta_{ij} + \Delta_{ji}) \stackrel{!}{=} 0 \quad , \quad \tilde{\mathcal{H}}_0^{\text{ADM}} \stackrel{!}{=} 0 \quad (\text{A10})$$

to determine the six fields \mathcal{K}_{ij} .

In a next step, we will now solve the Hamiltonian constraint (A5) explicitly. At any space-point $x \in \mathcal{M}_3$, eq. (A5) describes a five dimensional hyperboloid in the six dimensional space spanned by \mathcal{K}_{ij} , as long as

$$\forall x \in \mathcal{M}_3 : R(x) \neq 2\Lambda \quad , \quad (\text{A11})$$

which will be assumed in the following. This five dimensional hyperboloid may be parameterized with help of a stereographic projection, hence the general solution of the Hamiltonian constraint can be written in the form

$$\mathcal{K}^i_j = \frac{\sqrt{R-2\Lambda}}{1-\text{Tr}\mathcal{Q}^2} \left[\frac{1+\text{Tr}\mathcal{Q}^2}{\sqrt{6}} \delta_j^i + 2\mathcal{Q}^i_j \right] \quad , \quad \text{Tr}\mathcal{Q}^2 \neq 1 \quad , \quad (\text{A12})$$

where \mathcal{Q} is a symmetric, *traceless* matrix. Matrices \mathcal{Q} with $\text{Tr}\mathcal{Q}^2 = 1$ correspond to coordinate singularities of the stereographic projection, and thus have to be excluded in (A12). Inserting the general solution (A12) of $\tilde{\mathcal{H}}_0^{\text{ADM}} = 0$ into the first of eqs. (A10), we arrive at *five* equations for the *five* fields \mathcal{Q}^i_j , which remain to be determined.

We now want to argue that the effective set of partial differential equations obtained this way is soluble with respect to \mathcal{Q}^i_j . Let us therefore consider a background solution $\bar{\mathcal{Q}}^i_j$ of these equations, which we assume to be known for sufficiently simple parameter fields \bar{e}^i_a and Λ .¹³ Under infinitesimal perturbations of the parameter fields \bar{e}^i_a and Λ , the new solution \mathcal{Q}^i_j will differ from the background solution $\bar{\mathcal{Q}}^i_j$ by an infinitesimal amount

$$\mathcal{Q}^i_j = \bar{\mathcal{Q}}^i_j + \epsilon \cdot \mathcal{Q}'^i_j + \mathcal{O}(\epsilon^2) \quad , \quad (\text{A13})$$

and in the following it will be sufficient to show that the fields \mathcal{Q}'^i_j exist to any given background solution $\bar{\mathcal{Q}}^i_j$. Inserting the perturbation ansatz (A13) into $\Delta_{(ij)} = 0$, we arrive at five *linear* partial differential equations $\Delta'_{(ij)} = 0$ in $\mathcal{O}(\epsilon)$ determining the fields \mathcal{Q}'^i_j . To show that these equations are soluble with respect to \mathcal{Q}'^i_j , we will restrict ourselves to a discussion of the *symbol* of $\Delta'_{(ij)} = 0$, which we will show to be *elliptic* (cf. [45]). The symbol $\sigma(\mathbf{k})$ of a linear differential operator is obtained by computing the action on a Fourier mode

$$\mathcal{Q}'^i_j(x) = \hat{\mathcal{Q}}^i_j(\mathbf{k}) \cdot e^{i\mathbf{k}_\ell x^\ell} \quad , \quad (\text{A14})$$

in leading order of the wavevector \mathbf{k} . For the operator $\Delta'_{(ij)}$ under study, we obtain

$$\begin{aligned} \sigma_{ij}(\Delta'_{(mn)}; \mathbf{k}) = & -2 \frac{\sqrt{R-2\Lambda}}{(1-\text{Tr}\bar{\mathcal{Q}}^2)^2} \left[\sqrt{6} k_i k_j \bar{\mathcal{Q}}^{mn} \hat{\mathcal{Q}}_{mn} - |\mathbf{k}|^2 \left((1-\text{Tr}\bar{\mathcal{Q}}^2) \hat{\mathcal{Q}}_{ij} \right. \right. \\ & \left. \left. + \sqrt{\frac{2}{3}} \bar{\mathcal{Q}}^{mn} (h_{ij} + \sqrt{6} \bar{\mathcal{Q}}_{ij}) \hat{\mathcal{Q}}_{mn} \right) \right] \quad . \end{aligned} \quad (\text{A15})$$

The symbol $\sigma(\mathbf{k})$ is called elliptic, if it has a *trivial* kernel for $\mathbf{k} \neq \mathbf{0}$. Then the linear differential operator is invertable in the leading derivative order, and solutions of the linear differential equations will exist. To prove the ellipticity of the symbol (A15), it remains to be shown that the linear equations

$$\sqrt{6} q n_i n_j = \sqrt{\frac{2}{3}} q (h_{ij} + \sqrt{6} \bar{\mathcal{Q}}_{ij}) + (1-\text{Tr}\bar{\mathcal{Q}}^2) \hat{\mathcal{Q}}_{ij} \quad (\text{A16})$$

have only the trivial solution $\hat{\mathcal{Q}}_{ij} = 0$ for $\mathbf{n} \neq \mathbf{0}$, where we have introduced the abbreviations

$$q := \bar{\mathcal{Q}}^{ij} \hat{\mathcal{Q}}_{ij} \quad , \quad \mathbf{n} := \frac{\mathbf{k}}{|\mathbf{k}|} \Rightarrow |\mathbf{n}| = 1 \quad . \quad (\text{A17})$$

¹³ Explicit solutions \mathcal{A}_{ia} of the saddle-point eqs. (4.10), which correspond to the fields \mathcal{Q}^i_j via (A1) and (A12), are in fact known for various *homogeneous* 3-manifolds, such as Bianchi-type IX manifolds, cf. [24].

Contracting eqs. (A16) with $\bar{\mathcal{Q}}^{ij}$, we obtain the necessary implication

$$q \left(1 + \text{Tr} \bar{\mathcal{Q}}^2 - \sqrt{6} \bar{\mathcal{Q}}^{ij} n_i n_j \right) \stackrel{!}{=} 0 , \quad (\text{A18})$$

i.e. if we can show that the bracket in (A18) is different from zero, eq. (A18) implies $q = 0$, and therefore $\hat{\mathcal{Q}}_{ij} = 0$ via (A16), so the ellipticity of $\sigma(\mathbf{k})$ according to (A15) would have been proven.

It now follows from a simple estimate for symmetric matrices $\bar{\mathcal{Q}}$ that the vanishing of the bracket in (A18) implies¹⁴

$$1 + \sum_{i=1}^3 \bar{\mathcal{Q}}_i^2 \leq \sqrt{6} \max_{i=1}^3 \{ \bar{\mathcal{Q}}_i \} \quad (\text{A19})$$

where the $\bar{\mathcal{Q}}_i$ denote the three eigenvalues of the matrix $\bar{\mathcal{Q}}$. Since $\bar{\mathcal{Q}}$ is traceless, these three eigenvalues may be parameterized by

$$\bar{\mathcal{Q}}_j = \sqrt{\frac{2}{3}} \varrho \cos \left(\theta + \frac{2\pi j}{3} \right) , \quad j \in \{1, 2, 3\} \quad \text{with} \quad \varrho \geq 0 , \quad 0 \leq \theta < 2\pi . \quad (\text{A20})$$

Then the relation (A19) takes the form

$$1 + \varrho^2 \leq 2 \varrho \quad \Leftrightarrow \quad (1 - \varrho)^2 \leq 0 , \quad (\text{A21})$$

and is obviously only satisfied for $\varrho = 1$. Moreover, because of the identity $\text{Tr} \bar{\mathcal{Q}}^2 = \varrho^2$, the particular value $\varrho = 1$ corresponds to the coordinate singularity of the stereographic projection used in (A12), and is hence not permitted by construction. Thus the relation (A19) has been brought to a contradiction, and we conclude that the bracket in (A18) cannot vanish, what finishes our proof of the ellipticity of the symbol $\sigma(\mathbf{k})$ given in (A15).

Summarizing our results, we have shown that the set of linear partial differential equations $\Delta'_{(ij)} = 0$, which determines the fields \mathcal{Q}^i_j , is elliptic, and therefore soluble in leading derivative order. It follows, that the solutions \mathcal{Q}^i_j of the nonlinear set of equations $\Delta_{(ij)} = 0$ continue to exist under infinitesimal perturbations of the parameter fields \tilde{e}^i_a and Λ . Therefore, solutions \mathcal{K}_{ij} of (A7), and also solutions \mathcal{A}_{ia} of the saddle-point equations (4.10) can be obtained via (A12) and (A1) for a wide range of parameter fields \tilde{e}^i_a and Λ , as long as the only restriction $R \neq 2\Lambda$ met in (A11) is satisfied.

2. Divergence-free triads in the limit $\Lambda \rightarrow 0$

In this section we want to discuss how suitable *flat* gauge-fields \mathcal{A}_{ia} may be used to construct divergence-free triads $\vec{\mathcal{d}}_a$ of a given Riemannian 3-manifold $(\mathcal{M}_3, \mathbf{h})$. Such a flat gauge-field on \mathcal{M}_3 can be obtained by pursuing any *fixed* solution $\mathcal{A}_{ia}[\tilde{e}^i_a, \Lambda]$ of the saddle-point equations (4.10) in the limit $\Lambda \rightarrow 0$. Using the arguments of section A 1, this will be possible for 3-manifolds with $R(x) \neq 0$. By virtue of (2.25), the corresponding gauge-field \mathcal{A}_{ia} will not only be flat, but it will in addition satisfy the three Gauß-constraints

$$\mathcal{D}_i \tilde{e}^i_a \equiv \partial_i \tilde{e}^i_a + \varepsilon_{abc} \mathcal{A}_{ib} \tilde{e}^i_c = 0 , \quad (\text{A22})$$

where \tilde{e}^i_a is a fixed but arbitrary triad of the 3-metric \mathbf{h} .

Let us now consider the *parallel transport* associated with the gauge-field \mathcal{A}_{ia} : Given a vector $\vec{v}(0) = v_{a,0} \vec{e}_a$ at a point P_0 of \mathcal{M}_3 , and a curve $\mathcal{C} : x^i = f^i(u)$, $0 \leq u \leq 1$, connecting P_0 with a second point P_1 , we define a vector-field $\vec{v}(u)$ along \mathcal{C} by solving the equations of parallel transport,

$$\frac{\mathcal{D}v_a}{\mathcal{D}u} := \frac{\partial v_a}{\partial u} + \varepsilon_{abc} \frac{\partial f^i}{\partial u} \mathcal{A}_{ib} v_c \stackrel{!}{=} 0 \quad , \quad v_a(0) \stackrel{!}{=} v_{a,0} . \quad (\text{A23})$$

Since the gauge-field \mathcal{A}_{ia} is flat, the resulting vector $\vec{v}(1)$ at the Point P_1 does *not* depend on the particular choice

¹⁴ Here and in the following, we have to restrict ourselves to *real-valued* matrices $\bar{\mathcal{Q}}$, which correspond to real or complex solutions \mathcal{A}_{ia} of the saddle-point equations (4.10) via (A12) and (A1) in the two different cases $R > 2\Lambda$ or $R < 2\Lambda$, respectively.

of \mathcal{C} (cf. [46]), i.e. if we restrict ourselves to the case of *simply connected* manifolds \mathcal{M}_3 in the following, the parallel transport of $\vec{v}(0)$ along arbitrary curves $\mathcal{C} \subset \mathcal{M}_3$ will define a well-defined *vector-field* $\vec{v}(x)$ on \mathcal{M}_3 . By construction, this vector-field $\vec{v}(x)$ turns out to be covariantly constant with respect to \mathcal{A}_{ia} ,

$$\mathcal{D}_i v_a \equiv \partial_i v_a + \varepsilon_{abc} \mathcal{A}_{ib} v_c \equiv 0, \quad (\text{A24})$$

and, as a consequence of eq. (A22), the vector-field $\vec{v}(x)$ is in addition *divergence-free*,

$$\vec{\nabla} \cdot \vec{v} \equiv \frac{1}{\sqrt{h}} \mathcal{D}_i (v_a \tilde{e}^i_a) = \frac{1}{\sqrt{h}} \left(\underbrace{\mathcal{D}_i v_a}_0 \tilde{e}^i_a + v_a \underbrace{\mathcal{D}_i \tilde{e}^i_a}_0 \right) = 0. \quad (\text{A25})$$

Moreover, it follows from eq. (A24) that the parallel transport according to (A23) conserves the scalar product of two vectors \vec{v} and \vec{w} :

$$\partial_i (\vec{v} \cdot \vec{w}) \equiv \mathcal{D}_i (v_a w_a) = \underbrace{\mathcal{D}_i v_a}_0 w_a + v_a \underbrace{\mathcal{D}_i w_a}_0 = 0. \quad (\text{A26})$$

From eqs. (A25) and (A26) it is then obvious that a divergence-free *triad* $\vec{d}_a(x)$ of the Riemannian 3-manifold $(\mathcal{M}_3, \mathbf{h})$ can be constructed by choosing three orthonormal vectors \vec{d}_a at a point P_0 , and parallel-propagating these vectors along arbitrary curves $\mathcal{C} \subset \mathcal{M}_3$. Since the only freedom in this construction arises from the choice of \vec{d}_a at a single point P_0 , this divergence-free triad $\vec{d}_a(x)$ associated with the flat gauge-field \mathcal{A}_{ia} turns out to be unique up to *global* rotations.

APPENDIX B: THE VACUUM STATE ON BIANCHI-TYPE IX HOMOGENEOUS MANIFOLDS

In this appendix we want to discuss the semiclassical vacuum state (4.50) in the special case of Bianchi-type IX homogeneous 3-manifolds. While one saddle-point contribution, the so-called “wormhole-state”, is given by the result (4.53), four further semiclassical vacuum states are known within the framework of the homogeneous Bianchi IX model [24,26]. In the inhomogeneous approach of the present paper, these additional states should correspond to topologically nontrivial divergence-free triads of Bianchi-type IX manifolds via (4.50). Such special triads can indeed be constructed from the divergence-free triads of the unit 3-sphere, which will be discussed first in section B 1. The divergence-free triads of Bianchi-type IX manifolds and the corresponding saddle-point contributions to the vacuum Chern-Simons state will then be given in section B 2.

1. Divergence-free triads of the unit 3-sphere

The 3-sphere is a maximally symmetric 3-manifold with six killing-vectors $\vec{\xi}_a^\pm$, representing the commutator algebra

$$[\vec{\xi}_a^\pm, \vec{\xi}_b^\pm] = \pm 2 [abc] \vec{\xi}_c^\pm, \quad [\vec{\xi}_a^+, \vec{\xi}_b^-] = \vec{0} \quad (\text{B1})$$

of the symmetry group $SO(4) \cong SO(3) \times SO(3)$. From the second of these commutation relations it follows that the three vector-fields $\vec{\xi}_a^-$ are the left-invariant vector-fields to the killing-vectors $\vec{\xi}_a^+$, and vice versa, i.e. the metric tensor of the unit 3-sphere can be expanded in *both* of the two sets $\vec{\xi}_a^\pm$ with *spatially constant* coefficients. In particular, if we choose the normalization of $\vec{\xi}_a^\pm$ as in the first of eqs. (B1), the invariant vector fields $\vec{\xi}_a^\pm$ form automatically two different sets of *invariant triads* $\vec{\tau}_a^\pm := \vec{\xi}_a^\pm$ to the metric \mathbf{h} of the unit 3-sphere:

$$\vec{\tau}_a^+ \otimes \vec{\tau}_a^+ = \mathbf{h} = \vec{\tau}_a^- \otimes \vec{\tau}_a^-. \quad (\text{B2})$$

According to (B1) and (4.27), both invariant triads $\vec{\tau}_a^\pm$ have a *symmetric* structure matrix \mathbf{m} , and are thus *divergence-free* by virtue of eq. (4.51). Since they are triads to the same metric \mathbf{h} , they must be connected by a gauge-transformation $\mathbf{E} \in O(3)$:

$$\vec{\tau}_a^+ = E_{ab} \vec{\tau}_b^-. \quad (\text{B3})$$

The matrix \mathbf{E} has a spatially nontrivial dependence, and may of course be calculated explicitly in any given coordinate system on S^3 .¹⁵ However, in the following the explicit form of the rotation matrix \mathbf{E} will not be needed.

2. Divergence-free triads of Bianchi-type IX homogeneous manifolds

Anisotropic manifolds of Bianchi-type IX can be described by choosing an invariant triad of the unit 3-sphere, for example \vec{v}_a^+ , and rescaling this triad with three scale parameters $a_b > 0$:

$$\vec{v}_a := D_{ab} \vec{v}_b^+ \quad \text{with} \quad \mathbf{D}^{-1} := \text{diag}(a_1, a_2, a_3) . \quad (\text{B4})$$

Then \vec{v}_a is the invariant triad of a Bianchi-type IX manifold, and the metric tensor is given by $\mathbf{h} = \vec{v}_a \otimes \vec{v}_a$. In the general, anisotropic case, only three of the six vector-fields $\vec{\xi}_a^\pm$ discussed in section B 1 remain as killing-vectors of the 3-metric \mathbf{h} , namely the fields $\vec{\xi}_a^-$. We will assume that the invariant triad \vec{v}_a given in (B4) is positive-oriented. As pointed out in section IV C 1, this triad $\vec{d}_a^{(0)} := \vec{v}_a$ is automatically divergence-free, and gives rise to the “wormhole” saddle-point contribution (4.53) to the semiclassical vacuum state.

To find further, topologically nontrivial divergence-free triads \vec{d}_a of Bianchi-type IX metrics, let us try an ansatz of the form

$$\vec{d}_a = E_{ba} O_{bc} \vec{v}_c , \quad (\text{B5})$$

where $\mathbf{O} = (O_{ab}) \in SO(3)$ is assumed to be spatially constant.¹⁶ If we require the triad \vec{d}_a according to (B5), (B4) to be divergence-free, we arrive at three equations for the matrix \mathbf{O} ,

$$\vec{\nabla} \cdot \vec{d}_a = O_{bc} D_{cd} [\vec{v}_d^+, E_{ba}] \stackrel{!}{=} 0 . \quad (\text{B6})$$

The spatial derivatives of the matrix \mathbf{E} with respect to the vector-fields \vec{v}_a^+ can be calculated by inserting eqs. (B3) into eqs. (B1), and are given by

$$[\vec{v}_a^+, E_{bc}] = 2 \varepsilon_{abd} E_{dc} . \quad (\text{B7})$$

Therefore, the requirements (B6) can be simplified to the form

$$\varepsilon_{abc} O_{bd} D_{dc} \stackrel{!}{=} 0 , \quad (\text{B8})$$

i.e. the matrix \mathbf{O} has to be chosen in such a way that for any given diagonal matrix \mathbf{D} the matrix $\mathbf{O} \cdot \mathbf{D}$ is *symmetric*. The only four solutions $\mathbf{O} \in SO(3)$ of this problem turn out to be

$$\begin{aligned} \mathbf{O}^{(1)} &= \text{diag}(+1, -1, -1) & , & & \mathbf{O}^{(2)} &= \text{diag}(-1, +1, -1) , \\ \mathbf{O}^{(3)} &= \text{diag}(-1, -1, +1) & , & & \mathbf{O}^{(4)} &= \text{diag}(+1, +1, +1) , \end{aligned} \quad (\text{B9})$$

hence the ansatz (B5) gives exactly *four* further divergence-free triads of Bianchi-type IX homogeneous manifolds,

$$\vec{d}_a^{(\alpha)} = E_{ba} O_{bc}^{(\alpha)} \vec{v}_c \quad , \quad \alpha \in \{1, 2, 3, 4\} . \quad (\text{B10})$$

We now wish to compute the semiclassical saddle-point contributions to the vacuum state (4.50), which correspond to the divergence-free triads $\vec{d}_a^{(\alpha)}$, $\alpha \in \{1, 2, 3, 4\}$. Therefore we first need the winding numbers \hat{w} of these triads with respect to the Einstein-triad \vec{g}_a of Bianchi-type IX metrics. Since the Einstein-triad turns out to be given exactly by the invariant triad of the homogeneous 3-metric, $\vec{g}_a \equiv \vec{v}_a$, we have to calculate the Cartan-Maurer invariants (4.23) of the four rotation matrices

¹⁵For example, if we employ the Euler angles ψ, ϑ, φ as coordinates on the unit 3-sphere, the matrix \mathbf{E} turns out to be precisely the well-known Euler-matrix $\mathbf{E}(\psi, \vartheta, \varphi)$ (for a definition of the Euler-matrix, see e.g. [47]).

¹⁶At least in the isotropic case $a_1 = a_2 = a_3$, this ansatz gives the second divergence-free triad \vec{v}_a^- of the 3-sphere by virtue of (B3), if we simply choose $\mathbf{O} = \mathbf{1}$.

$$\mathbf{\Omega}^{(\alpha)} := \mathbf{E}^T \cdot \mathbf{O}^{(\alpha)} \quad , \quad \alpha \in \{1, 2, 3, 4\} . \quad (\text{B11})$$

This can be done *without* knowing the matrix \mathbf{E} in (B11) explicitly, because the spatial derivatives in (4.23) may be substituted by $\partial_j = \iota_{ja}^+ \cdot \vec{\tau}_a^+$, and then be eliminated with help of (B7), yielding

$$I(\mathbf{\Omega}^{(\alpha)}) = -8 \int d^3x \varepsilon_{abc} \iota_a^+ \wedge \iota_b^+ \wedge \iota_c^+ = -48 \mathcal{V} , \quad (\text{B12})$$

where $\mathcal{V} = 2\pi^2$ is the dimensionless volume of the unit 3-sphere. Since the constant I_0 in the definition (4.24) of the winding number has the numerical value $I_0 = 96\pi^2$ for manifolds with S^3 -topology (cf. [36]), it follows that the “absolute” winding numbers of the triads $\vec{d}_a^{(\alpha)}$, $\alpha \in \{1, 2, 3, 4\}$, are simply given by $\hat{w} = -1$.

To proceed in the computation of the semiclassical saddle-point contributions (4.50), we further have to evaluate the functional ϕ defined in (2.15) for the four divergence-free triads $\vec{e}_a = \vec{d}_a^{(\alpha)}$, $\alpha \in \{1, 2, 3, 4\}$. Inserting the triads (B10) into ϕ according to (2.15), we first recover the “wormhole”-exponent of (4.53), if the spatial derivative ∂_j acts on the invariant triad $\vec{\tau}_a$. In addition, we obtain a second term, which stems from the action of the derivative operator ∂_j on the spatially nontrivial matrix \mathbf{E} . This contribution can again be calculated by reexpressing the spatial derivative in terms of the vector-fields $\vec{\tau}_a^+$, and making use of eqs. (B7). In case of the divergence-free triad $\vec{d}_a^{(4)}$, we obtain the explicit result

$$\Psi_{\text{vac}}^{(4)} \stackrel{\mu \rightarrow \infty}{\propto} \Psi_{\text{vac}}^{(0)} \cdot \exp \left[\pm \frac{4\mathcal{V}}{\gamma\hbar} \left(-\frac{6}{\Lambda} + a_1 a_2 + a_2 a_3 + a_3 a_1 \right) \right] , \quad (\text{B13})$$

with $\Psi_{\text{vac}}^{(0)}$ given in (4.53). The saddle-point value (B13) is known as the “no-boundary” state from the homogeneous Bianchi IX model. Three further semiclassical saddle-point contributions to the vacuum state (4.50), which correspond to the remaining divergence-free triads $\vec{d}_a^{(\alpha)}$, $\alpha \in \{1, 2, 3\}$, are of the same form as $\Psi_{\text{vac}}^{(4)}$ given in (B13), but with two of the three scale parameters a_b replaced by their negatives. In the framework of the Bianchi IX model, the corresponding states were referred to as “asymmetric” states. We conclude that all five saddle-point values $\Psi_{\text{vac}}^{(\alpha)}$, $\alpha \in \{0, \dots, 4\}$, known for the homogeneous Bianchi IX model can be recovered within the inhomogeneous approach of the present paper by evaluating the state (4.50) for the five topologically inequivalent divergence-free triads $\vec{d}_a^{(\alpha)}$, $\alpha \in \{0, \dots, 4\}$, of Bianchi-type IX manifolds. Up to a Gaussian prefactor, which always lies hidden in the proportionality signs of eqs. (4.53), (B13), the results are of the same form as in [24,26].

However, as we have shown in [26,48], the four semiclassical saddle-point contributions $\Psi_{\text{vac}}^{(\alpha)}$, $\alpha \in \{1, 2, 3, 4\}$, are restricted to occur *simultaneously* for symmetry reasons. This can also be seen within the present, inhomogeneous approach, since the four divergence-free triads $\vec{d}_a^{(\alpha)}$, $\alpha \in \{1, 2, 3, 4\}$, all have the same winding number, and thus should enter into the value of the Chern-Simons state with the same topological right. We conclude that, in agreement with discussions of the non-diagonal Bianchi IX model, only *two* independent values of the vacuum Chern-Simons state are found for Bianchi-type IX manifolds.

APPENDIX C: A NON-FLAT 4-METRIC GENERATED BY THE VACUUM STATE

We now want to give special solutions of the vacuum evolution equations (4.55), such that the associated semiclassical 4-geometries satisfy neither the “no-boundary” condition proposed by Hartle and Hawking [38–40], nor the condition of asymptotical flatness in the limit of large scale parameters a_{cos} .¹⁷ Let us therefore consider the class of 3-metrics

$$\mathbf{h} = \vec{\tau}_a \otimes \vec{\tau}_a , \quad (\text{C1})$$

where the triad vector-fields $\vec{\tau}_a = \iota_a^i \partial_i$ are given by

$$\vec{\tau}_1 = \frac{1}{a_1} \partial_1 \quad , \quad \vec{\tau}_2 = \frac{1}{a_2} \partial_2 \quad , \quad \vec{\tau}_3 = \frac{1}{a_3} (\partial_3 + x^2 \partial_1 + x^1 \partial_2) . \quad (\text{C2})$$

¹⁷Here we assume the vacuum limit $\kappa \rightarrow 0$ to be realized by considering a sufficiently small value for the cosmological constant Λ . Then it will be possible to take the cosmological scale parameter a_{cos} arbitrarily large at the same time, cf. eq. (4.4).

The scale-parameters a_b in (C2) are assumed to be spatially constant, and the triad $\vec{\tau}_a$ is taken to be positive-oriented. Then the structure matrix \mathbf{m} introduced in (4.27) takes the spatially constant form

$$\mathbf{m} = \text{diag} \left[\frac{a_1}{a_2 a_3}, -\frac{a_2}{a_3 a_1}, 0 \right], \quad (\text{C3})$$

i.e. the triad $\vec{\tau}_a$ is the invariant triad of a spatially homogeneous 3-manifold, which can be classified to be of Bianchi-type VI₋₁. Since the structure matrix \mathbf{m} according to (C3) is symmetric, it follows directly from eq. (4.51) that the invariant triad $\vec{\tau}_a$ is divergence-free. The Killing-vectors of the 3-metric (C1) must commute with the $\vec{\tau}_a$ and are given by

$$\vec{\xi}_1 = \cosh x^3 \partial_1 + \sinh x^3 \partial_2, \quad \vec{\xi}_2 = \sinh x^3 \partial_1 + \cosh x^3 \partial_2, \quad \vec{\xi}_3 = \partial_3. \quad (\text{C4})$$

They may be used to compactify the 3-manifold \mathcal{M}_3 with the metric (C1) in the three $\vec{\xi}_a$ -directions, giving rise to a manifold with the nontrivial topology $S^1 \times T^2$. The compactified 3-manifold will then have a finite volume $V = \mathcal{V} a_1 a_2 a_3$, where the value of $\mathcal{V} > 0$ depends on the particular choice of the compactification.

We are now interested in the semiclassical 4-geometries being generated by the evolution equations (4.55) in case of the divergence-free triad $\vec{d}_a = \vec{\tau}_a$. If we allow for an arbitrary lapse-function N , they read

$$\frac{d}{d\tau} \tilde{\tau}^i{}_a = \pm N \tilde{\varepsilon}^{ijk} \partial_j \iota_{ka}. \quad (\text{C5})$$

For the three-metric (C1) under study, eqs. (C5) take the form

$$\frac{d}{d\tau} \sigma_1 = \mp N \sqrt{\frac{\sigma_2 \sigma_3}{\sigma_1}}, \quad \frac{d}{d\tau} \sigma_2 = \pm N \sqrt{\frac{\sigma_3 \sigma_1}{\sigma_2}}, \quad \frac{d}{d\tau} \sigma_3 = 0, \quad (\text{C6})$$

where we have introduced the new variables

$$\sigma_1 := a_2 a_3, \quad \sigma_2 := a_3 a_1, \quad \sigma_3 := a_1 a_2. \quad (\text{C7})$$

Choosing the lapse-function N as

$$N = \mp \frac{1}{2} (\sigma_1 \sigma_2 \sigma_3)^{-1/2}, \quad (\text{C8})$$

the set of eqs. (C6) is easily integrated and has the general solution

$$\sigma_1(\tau) = \sqrt{\tau_0 + \tau}, \quad \sigma_2(\tau) = \sqrt{\tau_0 - \tau}, \quad \sigma_3(\tau) \equiv \sigma_3 = \text{const.} \quad ; \quad |\tau| < \tau_0. \quad (\text{C9})$$

Here we have chosen $\tau = 0$ such that $\sigma_1(0) = \sigma_2(0)$, so only two integration constants $\tau_0 > 0$ and $\sigma_3 > 0$ remain in (C9).

In order to prove that the 4-geometry according to (C9) is non-flat, it is *not* sensible to compute the 4-dimensional Ricci- or Einstein-tensor, since these quantities vanish identically by construction, so we will consider the nontrivial components ${}^4R^{0i}{}_{0j}$ of the 4-dimensional Riemann-tensor instead. For a vanishing shift-vector $N^i = 0$, they are given by

$${}^4R^{0i}{}_{0j} = -\frac{1}{N} \frac{d}{dt} K^i{}_j + K^i{}_k K^k{}_j, \quad (\text{C10})$$

with $K^i{}_j$ being the usual extrinsic curvature tensor. With help of the triad (C2), we may convert the spatial indices of ${}^4R^{0i}{}_{0j}$ into internal indices a, b , to obtain

$$\mathcal{R}_{ab} := \iota_{ia} \iota^j{}_b {}^4R^{0i}{}_{0j}. \quad (\text{C11})$$

For the metric (C1), (\mathcal{R}_{ab}) is a diagonal matrix with

$$\mathcal{R}_{33} = -\frac{1}{N a_3} \frac{d}{d\tau} \left(\frac{1}{N} \frac{da_3}{d\tau} \right), \quad (\text{C12})$$

and analogous expressions for $\mathcal{R}_{11}, \mathcal{R}_{22}$. Making use of the evolution eqs. (C6), we can eliminate the τ -derivatives in (C12) to arrive at

$$\mathcal{R}_{33} = \frac{\sigma_1 \sigma_2 \sigma_3}{2} \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)^2, \quad (\text{C13})$$

and inserting the general solution (C9) into (C13), we find

$$\mathcal{R}_{33} = \sigma_3 \tau_0^2 (\tau_0^2 - \tau^2)^{-3/2} > 0. \quad (\text{C14})$$

Thus we have found a component of the Riemann-tensor, which is nonzero for all times τ , $|\tau| < \tau_0$, so the semiclassical 4-geometries obtained by evolving the initial 3-geometries (C1) are *nowhere* flat. Moreover, the semiclassical 4-geometries do *not* satisfy the “no-boundary” condition: While the cosmological scale parameter

$$a_{\text{cos}} = \mathcal{V}^{1/3} (\sigma_1 \sigma_2 \sigma_3)^{1/6} = \mathcal{V}^{1/3} \sigma_3^{1/6} (\tau_0^2 - \tau^2)^{1/12} \quad (\text{C15})$$

vanishes only at the timelike borders $|\tau| \rightarrow \tau_0$ of the semiclassical space-time manifolds, the corresponding curvature components \mathcal{R}_{33} at the same time are tending to $+\infty$. Consequently, the semiclassical 4-manifolds are *not* regular or compact for vanishing scale parameters a_{cos} .

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